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THERE ARE EXACTLY 172 CONNECTED Q–INTEGRAL GRAPHS UP TO 10 VERTICES[1]

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Abstract. A graph is called Q–integral if its signless Laplacian spectrum consists entirely of integers. We establish that there are exactly 172 connected Q–integral graphs up to 10 vertices. Pictures or adjacency matrices of those graphs, their Q–spectra, some data and comments are given. In addition, we present the connected graphs of the smallest order (which are neither regular nor complete bipartite) being integral in the sense of each of the following three spectra: usual one (related to the adjacency matrix), Laplacian and signless Laplacian.

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1. Introduction

Let G be a simple graph with adjacency matrix A $(= A_G)$. The eigenvalues and the spectrum of A are also called the *eigenvalues* and the *spectrum* of G, respectively. A graph whose spectrum consists entirely of integers is called an *integral graph*. If we consider a matrix $L = D - A$ instead of A, where D is the diagonal matrix of vertex–degrees (in G), we get the *Laplacian eigenvalues* and the *Laplacian spectrum*, while in the case of matrix $Q = D+A$ we get the *signless Laplacian eigenvalues* and the *signless Laplacian spectrum*, respectively. For short, the signless Laplacian eigenvalues and the signless Laplacian spectrum will be called the *Q–eigenvalues* and the *Q–spectrum*, respectively. A graph whose Laplacian (resp. signless Laplacian) spectrum consists entirely of integers is called an *L–integral* (resp. *Q–integral*) graph.

The integral and L–integral graphs are well studied (see [5], where a survey is given). On the other hand, the graphs with integral Q–spectrum are studied in exactly one forthcoming paper [6], so fare. Since the matrix Q is positive semidefinite, the Q–spectrum consists of non–negative values. Furthermore, the least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite; in this case 0 is a simple eigenvalue (see [3], Proposition 2.1).

Recall that if G is a regular graph which is integral in the sense of any of spectra mentioned above, then it has integral the other two spectra (cf. [3], Section 3), as well. In particular, the complete graphs make an infinite series of

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graphs having all three spectra integral. Also, if G is a bipartite graph then its L–spectrum and Q–spectrum coincide (the proof can be found in many places, see [4], for example), and therefore every bipartite graph is L–integral if and only if it is Q–integral.

In Section 2, connected Q–integral graphs up to 10 vertices, their Q–spectra and some additional data are given. Some comments are given in Section 3. Finally, in Section 4, we emphasize two connected graphs (which are neither regular nor complete bipartite) of the smallest order being integral, L–integral and Q–integral, at the same time.

2. Data on Q–integral graphs

The connected Q–integral graphs up to 5 vertices can easily be found (even by hand). Those graphs are: K_1 (order 1), K_2 (order 2), $K_{1,2}$, K_3 (order 3), $K_{1,3}$, $K_{2,2}$, K_4 (order 4), $K_{1,4}$, $K_{2,3}$, K_5 (order 5). By direct computing, one can obtain the Q–spectra of these graphs. Using the computer we obtain all connected Q–integral graphs on n ($6 \leq n \leq 10$) vertices.

The numbers q_n of connected Q–integral graphs with n vertices are given for $n = 1, ..., 10$ in the following table.

n	1	2	3	4	5	6	7	8	9	10
q_n	1	1	2	3	3	13	14	18	26	91

Table 1.

Summing up the last row we get that there are exactly 172 connected Q–integral graphs up to 10 vertices.

In Fig. 1–Fig. 4, connected Q–integral graphs of order n ($6 \leq n \leq 9$) are given, while connected Q–integral graphs of order 10 are displayed in List 1. Graphs in List 1 are represented in the following form

$$\text{No} \quad a_{1,2} \cdots a_{1,10} \quad a_{2,3} \cdots a_{2,10} \quad \cdots \quad a_{8,9} a_{8,10} \quad a_{9,10} \ ,$$

where No is the identification number of the corresponding graph, while the rest is the upper triangle of its (adjacency) matrix. All the presented graphs (in figures and list) are ordered by the number of edges and by their Q–spectra.

The data of obtained graphs are given in Lists 2–6. Each row contains the identification number of the corresponding graph, the number of edges, the Q–spectrum (where the exponents stand for the multiplicities of the eigenvalues) and the additional information contained in the last two columns. Firstly, a short description (provided by unions (+) and/or joins (∇) of regular and/or complete bipartite graphs) of some graphs is given, while in the last column any of the marks A, L, C is present if the corresponding graph is integral, L–integral or if its complement is Q–integral and disconnected, respectively. The complement which is Q–integral and connected is present in the same list, and in that case its identification number is given in the brackets. Also, the identification numbers of cospectral graphs (in the sense of Q–spectrum) are given in bold. Note that cospectral graphs are placed one–next–to–other in Lists.

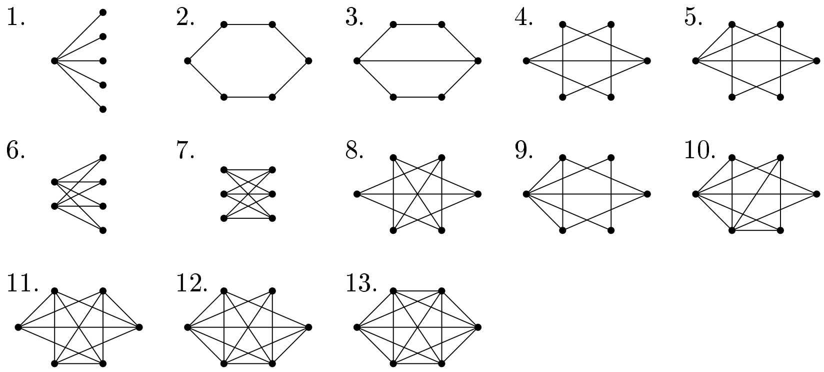


Fig. 1

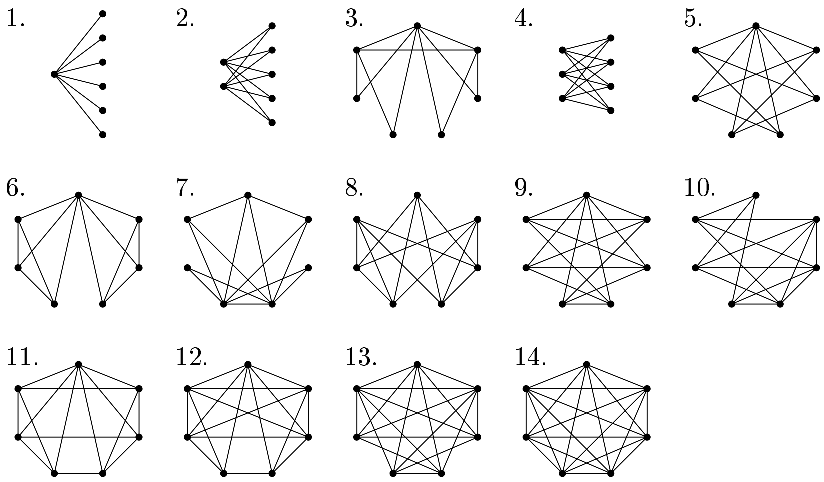


Fig. 2

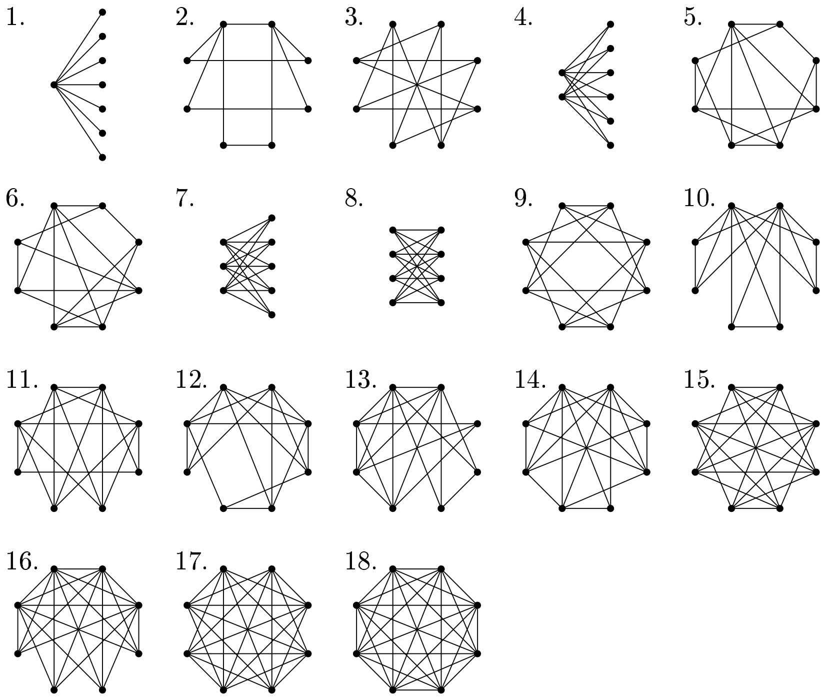
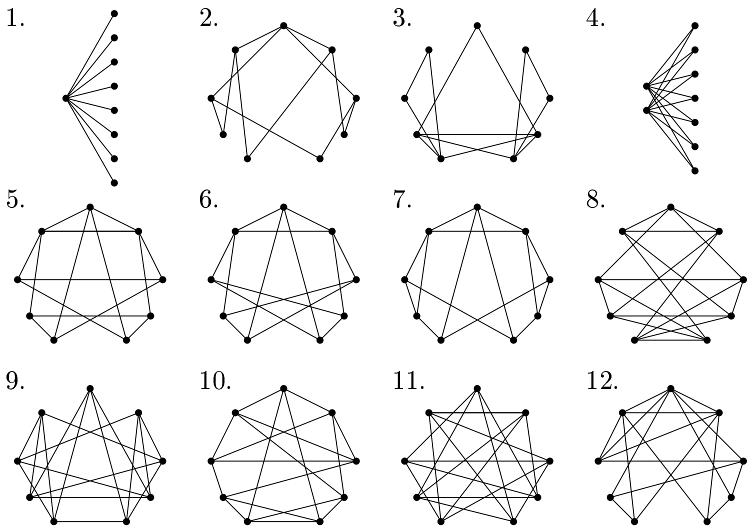


Fig. 3



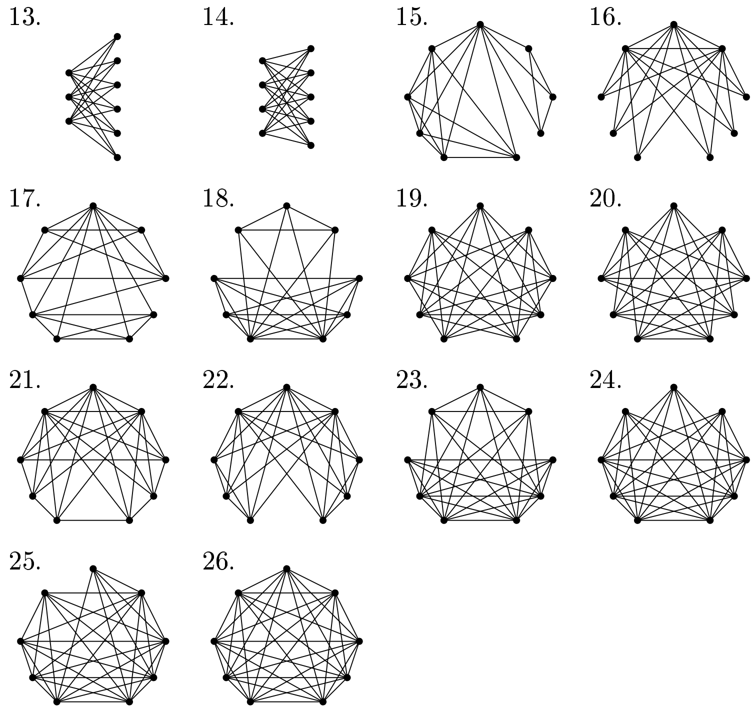


Fig. 4

LIST 1. Q–INTEGRAL GRAPHS WITH 10 VERTICES

1.	000000001	00000001	0000001	000001	00001	0001	001	01	1
2.	000001100	00001010	0001001	000110	00101	0011	000	00	0
3.	000100010	00010010	0001010	000110	00001	0001	001	01	1
4.	001001001	00101010	0011100	000110	00101	0011	000	00	0
5.	000011100	00011100	0010011	001011	00111	0000	000	00	0
6.	001001100	00101001	0010101	001010	00110	0011	000	00	0
7.	000100100	00010010	0001001	000111	00100	0010	001	11	1
8.	000011101	00010010	0001111	001011	00111	0000	000	00	0
9.	000011110	00011101	0001111	000011	00011	0000	000	00	0
10.	000010110	00010110	0001010	001001	00101	1001	001	10	0
11.	000001100	00001100	0001100	000011	00011	0011	011	11	0
12.	000001100	00001010	0001010	000101	00101	0011	011	11	0
13.	000000011	00000011	0000011	000011	00011	0011	011	11	0
14.	001001001	00101001	0011001	000110	00110	0110	010	01	0
15.	001001001	00101010	0011010	000110	00101	0101	010	01	0
16.	000100011	00010010	0001101	001101	00011	0010	001	01	1

17.	000010011	00010011	0001111	001111	01111	0000	000	00	0
18.	000001101	00001101	0001011	001011	00111	0111	000	00	0
19.	000100011	00011110	0010101	001101	00011	0001	001	10	0
20.	001001110	00100101	0010011	001001	00101	0011	001	10	0
21.	000011011	00000111	0000111	000011	00011	1000	000	11	1
22.	000011001	00010111	0010111	001111	01111	0000	000	00	0
23.	000011110	00011101	0011011	010111	01111	0000	000	00	0
24.	000110110	00001111	0001001	001001	10110	0110	001	01	1
25.	000011111	00011111	0011111	000111	00111	0000	000	00	0
26.	000011110	00011110	0000011	000011	00011	1101	101	01	1
27.	000000111	00000111	0000111	000111	00111	0111	111	00	0
28.	000110001	00001111	0001111	001111	10001	0001	001	01	1
29.	001010001	00101101	0001111	010001	00111	0001	011	01	1
30.	000100011	00010011	0001011	000111	00011	0011	011	11	1
31.	000011111	00011111	0011011	010111	01111	0000	000	00	0
32.	010101011	01010111	0101010	010101	01010	0101	010	01	0
33.	000011110	00011110	0000111	000111	00111	1001	001	01	1
34.	000101100	00010011	0001111	001111	01100	0011	011	11	0
35.	001100001	00011111	0011111	100001	00001	0111	111	00	0
36.	001001111	00100111	0010111	001111	00111	0111	000	00	0
37.	010101011	01010101	0101011	010101	01011	0101	010	01	0
38.	000111100	00011111	0001111	001111	01111	0011	001	00	0
39.	000001111	00001111	0001111	001111	01111	1111	000	00	0
40.	001110011	00111010	0011111	001101	00111	0010	011	01	0
41.	001001111	00111010	0110101	001111	10011	0011	010	01	0
42.	001001111	00111001	0110110	001111	10111	0001	001	10	0
43.	010101011	01010111	0101011	010111	01010	0101	010	01	0
44.	010101111	01010111	0101001	010111	01001	0110	001	10	0
45.	000011111	00011111	0011111	011111	11111	0000	000	00	0
46.	010111010	01011101	0101110	010111	01011	0101	010	01	0
47.	000100111	00011111	0011111	011111	00111	0011	000	00	1
48.	000010111	00001111	0001111	001111	00111	0111	111	00	0
49.	000101111	00010111	0000111	000111	01111	0111	111	00	0
50.	010101011	01010111	0101011	010101	01011	0101	011	01	0
51.	001101010	01011001	0111111	001111	00110	0101	011	11	0
52.	010101111	01011111	0101010	010101	01010	0101	011	11	0
53.	010101011	01010111	0101011	010111	01011	0111	010	01	0
54.	010111100	01011011	0101111	010111	01111	0011	100	00	1
55.	001101011	00111101	0011111	011101	00111	0011	001	01	1
56.	001100111	01011011	0111101	001101	01011	0111	001	01	1
57.	001001111	00110111	0110111	001111	11001	1001	001	01	1
58.	001101101	01011011	0110111	001101	01011	0111	001	01	1
59.	000111111	00000111	0000111	000111	11111	1111	111	00	0
60.	000010011	00010011	0001111	001111	01111	1111	011	11	1
61.	000111011	00110111	0101111	011111	00011	0011	011	11	0
62.	010101011	01010111	0101011	010111	01011	0111	011	11	0
63.	010111010	01110111	0101110	011111	01011	0101	011	01	1
64.	001001111	00101111	0011111	001111	01111	1111	010	01	0
65.	001111011	00111110	0111101	011111	00111	0010	001	11	1
66.	001111010	01011111	0111110	101011	00111	0101	001	11	1

67.	000111111	00111111	0111111	111111	00110	0101	011	00	0
68.	001111110	01011111	0111111	101011	00111	1110	001	01	0
69.	011011011	01111110	0110111	011101	01111	0110	011	01	0
70.	000111111	00111111	0111111	111111	01010	0101	010	01	0
71.	010111101	01011111	0110111	011111	01111	1010	010	01	0
72.	000011111	00011111	0011111	000111	00111	0111	111	11	1
73.	001111111	01011110	0111110	110111	01111	0001	001	11	1
74.	010111111	01101110	0111111	011110	10001	0001	111	11	1
75.	010111011	01110111	0101110	011111	01011	0111	011	01	1
76.	001100111	00011111	0011111	100111	00111	0111	111	11	1
77.	000011111	00011111	0001111	001111	01111	1111	011	11	1
78.	010111111	01000111	0111111	000111	11111	1111	000	11	1
79.	000111111	00111111	0111111	111111	00011	0011	011	11	1
80.	010111011	01110111	0101111	011111	01011	0111	011	11	1
81.	001101111	01011111	0111111	001111	01111	1111	011	11	0
82.	010111111	01111111	0101011	010111	01111	1111	011	11	0
83.	010101111	01011111	0101111	011111	01111	1111	011	11	0
84.	001111111	01111111	1111111	001111	01111	1111	010	01	0
85.	010111111	01111111	0111111	111111	01110	0111	011	01	0
86.	001111111	01111111	1111111	001111	01111	1111	111	11	1
87.	001111111	01111111	1111111	011111	00111	1111	111	11	1
88.	011111111	11110011	0111111	001111	11111	1111	111	11	1
89.	011101111	01111111	0111111	011111	01111	1111	111	11	1
90.	011111111	11111111	0111111	111111	01111	1111	011	11	0
91.	111111111	11111111	1111111	111111	11111	1111	111	11	1

LIST 2. DATA OF Q–INTEGRAL GRAPHS WITH 6 VERTICES

1.	5			6	1^4	0	$K_{1,5}$	L C
2.	6		4	3^2	1^2	0	C_6	A L C(8.)
3.	7	5	3^2	2	1	0		L
4.	7		5	4	2	1^3		
5.	8		6	4	2^2	1^2		
6.	8		6	4	2^3	0	$K_{2,4}$	L C
7.	9			6	3^4	0	$K_{3,3}$	A L C
8.	9		6	4	3^2	1^2	3–regular	A L C(2.)
9.	9		7	4	2^3	1	$(K_2 \dot{+} K_3)\nabla K_1$	L C
10.	11			8	4^2	2^3	$K_2\nabla 2K_2$	L C
11.	12			8	4^3	2^2	4–regular	A L C
12.	13		9	4^3	3	2	$(K_1 \dot{+} K_2)\nabla K_3$	L C
13.	15				10	4^5	K_6	A L C

LIST 3. DATA OF Q–INTEGRAL GRAPHS WITH 7 VERTICES

1.	6				7	1^5	0	$K_{1,6}$	$L\ C$
2.	10			7	5	2^4	0	$K_{2,5}$	$L\ C$
3.	11			8	4^2	2^2	1^2		
4.	12			7	4^2	3^3	0	$K_{3,4}$	$L\ C$
5.	12		8	4^2	3	2^2	1	$C_6 \nabla K_1$	$L\ C$
6.	12			8	5	3	2^4	$2K_3 \nabla K_1$	$L\ C$
7.	13	9	5	4	3	2^2	1		L
8.	14		8	5	4^2	3^2	1	4–regular	$A\ L\ C$
9.	15				9	4^5	1	$K_{3,3} \nabla K_1$	$L\ C$
10.	15				9	4^5	1	$(K_1 \dotplus K_3) \nabla 3K_1$	$L\ C$
11.	15			9	5	4^3	2^2	Cone(3–regular)	$L\ C$
12.	17			10	5^2	4^3	2		$L\ C$
13.	19			11	5^4	4	3	$C_4 \nabla K_3$	$L\ C$
14.	21					12	5^6	K_7	$A\ L\ C$

LIST 4. DATA OF Q–INTEGRAL GRAPHS WITH 8 VERTICES

1.	7				8	1^6	0	$K_{1,7}$	$L\ C$
2.	10	6	4	3^2	2	1^2	0		L
3.	12			6	4^3	2^3	0	3–regular	$A\ L\ C$(9.)
4.	12			8	6	2^5	0	$K_{2,6}$	$L\ C$
5.	15				8	4^5	1^2		
6.	15		8	5	4^3	2^2	1		
7.	15			8	5^2	3^4	0	$K_{3,5}$	$L\ C$
8.	16				8	4^5	0	$K_{4,4}$	$A\ L\ C$
9.	16			8	6	4^3	2^3	4–regular	$A\ L\ C$(3.)
10.	16			10	6	4^2	2^4	$3K_2 \nabla K_2$	$L\ C$
11.	17		9	5^2	4^3	2	1		
12.	17		9	5^2	4^3	2	1		
13.	18	10	6	5	4^2	3	2^2		
14.	19		10	6	5^2	4^2	2^2		C
15.	20			10	6^2	4^4	2	5–regular	$A\ L\ C$
16.	22			12	6^3	4^3	2	$4K_1 \nabla K_4$	$L\ C$
17.	24				12	6^4	4^3	6–regular	$A\ L\ C$
18.	28					14	6^7	K_8	$A\ L\ C$

LIST 5. DATA OF Q–INTEGRAL GRAPHS WITH 9 VERTICES

1.	8					9	1^7	0	$K_{1,8}$	L C
2.	12		6	4^2	3^2	2	1^2	0		L
3.	13		7	5	4	3	2^2	1^3		L
4.	14				9	7	2^6	0	$K_{2,7}$	L C
5.	15					7	4^5	1^3		L
6.	15			7	5	4^3	2^2	1^2		L
7.	15			7	5	4^3	2^2	1^2		
8.	18			8	5^3	4^2	2^2	1	4–regular	L C(11.)
9.	18					8	5^4	2^4	4–regular	L C(9.)
10.	18	8	6	5	4^2	3^2	2	1	4–regular	L C(10.)
11.	18			8	6	5^2	3^2	2^3	4–regular	L C(8.)
12.	18			9	6	4^4	2^2	1		
13.	18				9	6^2	3^4	0	$K_{3,6}$	L C
14.	20				9	5^3	4^4	0	$K_{4,5}$	L C
15.	21				11	7	4^5	2^2	$(K_3 \dot{+} K_5)\nabla K_1$	L C
16.	21				12	7^2	3^5	1	$6K_1\nabla K_3$	L C
17.	22			11	7	6	4^2	3^4		
18.	24				12	7^2	4^4	3^2	$(K_3 \dot{+} K_4)\nabla K_2$	L C
19.	27			12	7	6^4	4^2	3	6–regular	A L C
20.	27					12	6^6	3^2	6–regular	A L C
21.	27			13	7^2	6^2	4^3	3	$C_6\nabla K_3$	L C
22.	27					13	7^3	4^5	$(K_3 \dot{+} K_3)\nabla K_3$	L C
23.	30				14	7^4	5^2	4^2	$(K_2 \dot{+} K_3)\nabla K_4$	L C
24.	33				15	7^5	6^2	4	$3K_1\nabla K_6$	L C
25.	33				15	7^5	6^2	4	$(K_1 \dot{+} K_3)\nabla K_5$	L C
26.	36						16	7^8	K_9	A L C

LIST 6. DATA OF Q–INTEGRAL GRAPHS WITH 10 VERTICES

1.	9					10	1^8	0	$K_{1,9}$	A L C
2.	12			5	4^3	2^2	1^3	0		L
3.	13		7	5	3^3	2	1^3	0		A L
4.	15					6	4^5	1^3	3–regular	A L C(69.)
5.	15	6	5	4^2	3^2	2^2	1	0	3–regular	A L C(71.)
6.	15			6	5	4^3	2^2	1^3	3–regular	A L C(68.)
7.	15			8	5^2	3	2^3	1^3		
8.	16	7	5	4^3	3	2^2	1	0		L
9.	16	7	5^2	4	3^2	2^2	1	0		L
10.	16		7	6	4^2	3^2	2	1^3		
11.	16			8	5^3	2^4	1	0		L
12.	16		8	6	5	4	2^3	1^3		
13.	16				10	8	2^7	0	$K_{2,8}$	A L C

14.	17		7	5^2	4^2	3	2^3	0		L
15.	17		7	6	4^3	3	2^2	1^2		
16.	17		9	6	4	3^3	2^2	1^2		
17.	18		8	5^2	4^2	3^3	1	0		
18.	18			8	5^3	3^3	2^2	0		L
19.	18		8	6	5	4^2	2^4	1		L
20.	18				8	6	5^2	2^6		
21.	18	10	6	5	4	3	2^3	1^2		
22.	19			8	5^2	4^4	2^2	0		L
23.	20				8	5^4	3^4	0	4–regular	$A\ L\ C(46.)$
24.	20	9	7	5	4^2	3^2	2^2	1		$C(47.)$
25.	20			9	5^5	3^2	2	0		L
26.	21		10	7	6	4^3	2^3	1		
27.	21				10	7^2	3^6	0	$K_{3,7}$	$L\ C$
28.	21			11	6	4^5	2^2	1		L
29.	21			11	6	5	4^3	2^4		L
30.	21				12	8	4^3	2^5	$4K_2\nabla K_2$	$A\ L\ C$
31.	22			9	5^5	4	3^2	0		L
32.	22			9	8	5	4	3^6		
33.	22		10	6^3	4	3^3	2	1		
34.	22				10	6^3	4^3	2^4		
35.	22		10	7	5^2	4^2	3	2^3		$C(36.)$
36.	23	10	6^3	5	4	3^2	2	1		$C(35.)$
37.	23			10	8	5	4^2	3^5		
38.	24		10	6^2	5^3	4^2	2	1		
39.	24				10	6^3	4^5	0	$K_{4,6}$	$L\ C$
40.	24	10	7	6	5^2	4^2	3	2^2		
41.	24			10	7^2	4^4	3^2	2		
42.	24			10	7^2	4^4	3^2	2		
43.	24			10	8	6	4^3	3^4		
44.	24			10	8	6	4^3	3^4		
45.	25					10	5^8	0	$K_{5,5}$	$A\ L\ C$
46.	25				10	8	5^4	3^4	5–regular	$A\ L\ C(23.)$
47.	25		11	6^3	5^2	4	3^2	1		$C(24.)$
48.	25		11	7^2	6	4^3	3^2	1		L
49.	25		11	7^2	6	4^3	3^2	1		L
50.	25			11	8	6	4^4	3^3		
51.	26		11	7^2	6	4^4	3	2		
52.	26			11	8	6^2	4^3	3^3		
53.	26			11	8	7	4^5	3^2		
54.	27			11	8	6	5^4	3^3		
55.	27			12	6^4	5^2	3^2	2	Cone(4–regular)	$L\ C$
56.	27					12	6^5	3^4	Cone(4–regular)	$L\ C$
57.	27	12	7	6^2	5^2	4^2	3	2	Cone(4–regular)	$L\ C$
58.	27			12	7	6^3	4^2	3^3	Cone(4–regular)	$L\ C$

59.	27		12	7^2	6	5^3	3^2	1		L
60.	27		13	8	6^3	4^2	3	2^2		C
61.	28			12	8	6^3	4^4	2		$L\ C$
62.	28					12	8^2	4^7	$(K_4 \dot{+} K_4)\nabla 2K_1$	$L\ C$
63.	29		12	8	6^3	5^2	4	3^2		
64.	29			12	8	6^4	4^3	2	$2K_2\nabla 3K_2$	$L\ C$
65.	29		12	7^2	6^2	5^2	4^2	2		
66.	29		12	7^2	6^2	5^2	4^2	2		
67.	30		12	7^2	6^3	5^2	4	2	6–regular	$A\ L\ C$
68.	30			12	7^3	6^2	4^3	3	6–regular	$A\ L\ C(6.)$
69.	30					12	7^4	4^5	6–regular	$A\ L\ C(4.)$
70.	30			12	8	6^3	5^4	2	6–regular	$A\ L\ C$
71.	30	12	8	7	6^2	5^2	4^2	3	6–regular	$A\ L\ C(5.)$
72.	30		14	8^2	6^2	5^2	3^2	2		L
73.	31		13	7^2	6^4	5	4	2		
74.	31			13	8	6^5	5	3^2		
75.	31	13	8	7	6^3	5	4^2	3		
76.	31		14	8^2	7	5^2	4^3	3	$(C_4 \dot{+} K_3)\nabla K_3$	$L\ C$
77.	31	14	8^2	7	6	5	4^3	2		$L\ C$
78.	32		14	8	7^2	6^3	4	3^2		
79.	33				14	8	6^7	2	$4K_1\nabla 4K_1\nabla K_2$	$L\ C$
80.	33				14	8^2	6^4	4^3		$L\ C$
81.	34		14	8^2	7^2	6	5^2	4^2	$C_4\nabla C_6$	$L\ C$
82.	34		14	8^2	7^2	6	5^2	4^2		$L\ C$
83.	34			14	8^3	6	5^4	4	$C_4\nabla 2K_3$	$L\ C$
84.	35			14	8	7^4	6^2	4^2	7–regular	$A\ L\ C$
85.	35		14	8^2	7^2	6^2	5^2	4	7–regular	$A\ L\ C$
86.	39			16	8^3	7^4	6	4	$3K_1\nabla 3K_1\nabla K_4$	$L\ C$
87.	39			16	8^3	7^4	6	4		$L\ C$
88.	39			16	8^3	7^4	6	4		$L\ C$
89.	39			16	8^4	7^2	6	5^2	$\overline{C_6}\nabla K_4$	$L\ C$
90.	40					16	8^5	6^4	8–regular	$A\ L\ C$
91.	45						18	8^9	K_{10}	$A\ L\ C$

3. Some comments

Based on the data on Q–integral graphs presented in this paper, several observations can be made. Firstly, we comment some facts concerning Q–integral graphs having connected Q–integral complements.

Among the graphs up to 5 vertices only K_1 and its complement (K_1, again) are both connected and Q–integral. There are just one pair of such graphs on 6 vertices, one pair on 8 vertices and one pair on 9 vertices. After K_1, the first self–complementary Q–integral graphs appear among the graphs of order 9 (there are two such graphs: No 9 and No 10 of Fig. 4). Finally, there are

5 pairs of such graphs on 10 vertices. Note that No 24 and No 47, as well as No 35 and No 36 of List 1, are the only two pairs of complementary Q–integral graphs which are not regular.

No two non–isomorphic connected Q–integral graphs on less than 7 vertices are cospectral (in the sense of Q–spectra). There is just one pair of such graphs on 7 vertices, one pair on 8 vertices (also, these two graphs are cospectral in the sense of L–spectra), two pairs on 9 vertices and 4 pairs (among them No 41 and No 42, as well as No 43 and No 44 of List 1, are also cospectral in the sense of L–spectra), and there is one triplet on 10 vertices. There are no two non–isomorphic graphs which are cospectral in the sense of both (usual one) spectra and Q–spectra, up to 10 vertices.

It is worth to mention that No 11 of Fig. 2 is a cone over No 8 of Fig. 1. Also, No 55, 56, 57 and 58 of List 1 are cones over No 8, 9, 10 and 11 of Fig. 4, respectively.

Now we have the following two lemmas.

Lemma 1. *Every complete bipartite graph is Q–integral.*

Proof. Recall that an arbitrary graph is Q–integral if and only if its line graph is integral (see, for example, [6], Section 2). Now, if $K_{m,n}$ is a complete bipartite graph, then its line graph is $K_m + K_n$, where $+$ denotes the sum of graphs. On the other hand, the spectrum of $K_m + K_n$ consists of all possible sums of the eigenvalues λ_i of K_m and λ_j of K_n (see [2], p. 70). The fact that each eigenvalue of a complete graph is integral completes the proof. □

Lemma 2. *A complete bipartite graph $K_{m,n}$ is integral, L–integral and Q–integral, at the same time if and only if mn is a perfect square.*

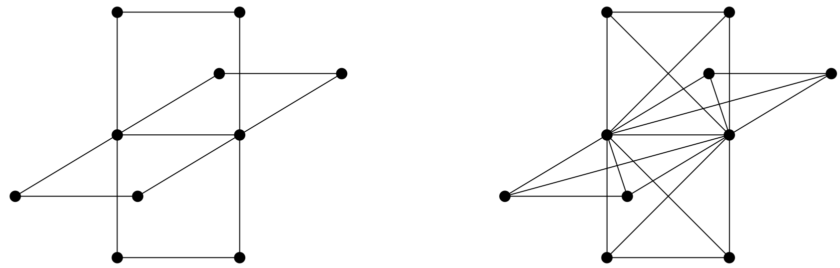
Proof. Recall that L–spectrum and Q–spectrum coincide for bipartite graphs and, due to Lemma 1, they are integral. Further, the spectrum of a complete bipartite graph $K_{m,n}$ contains the numbers $\sqrt{mn}$, $-\sqrt{mn}$ and $m+n-2$ numbers all equal to 0 (see [2], p. 72), which completes the proof. □

Exactly 42 connected graphs up to 10 vertices have all three spectra integral and 40 out of them are either regular or complete bipartite or both. The remaining two graphs are considered in the next section.

4. An additional data

There is no connected graph (which is neither regular nor complete bipartite) on less than 10 vertices being integral, L–integral and Q–integral. There are exactly two such graphs of order 10. They are No 3 and No 30 of List 1, and these graphs are depicted in Fig. 5, as well.

The first is a bipartite graph, and so its L–spectrum is the same as its Q–spectrum ($[7, 5, 3^3, 2, 1^3, 0]$), while its spectrum is $[3, 1^4, -1^4, -3]$. Its complement is a connected L–integral graph.



No 3 of List 1 No 30 of List 1

Fig. 5

The spectrum, L–spectrum and Q–spectrum of the second graph are $[5, 1^3, -1^5, -3]$, $[10^2, 4^4, 2^3, 0]$ and $[12, 8, 4^3, 2^5]$, respectively. Its complement is a disconnected integral, L–integral and Q–integral graph which consists of three components: two isolated vertices and a graph isomorphic to No 17 of Fig. 3.

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