Controllability of the multi-agent system modeled by the chain graphs with repeated degree

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***Abstract*—We consider the controllability of multi-agent dynamical systems modeled by a special class of bipartite graphs, called chain graphs. Our particular attention is focused on chain graphs that have one repeated degree. We derive properties of eigenvectors of graphs under consideration as well as some of their Laplacian spectra. On the basis of the obtained theoretical results, we determine the minimum number of leading agents that make the system in question controllable and locate them in the corresponding graph.**

***Index Terms*—Chain graph, Laplacian spectrum, Eigenvectors, Controllable dynamical system**

I. Introduction

Let $G = (V(G), E(G))$ be a simple graph (without loops or multiple edges) of *order* $n = |V(G)|$. By $A(G)$ we denote its $(0, 1)$-*adjacency matrix*. If $D(G)$ is the diagonal matrix of vertex degrees, then $L(G) = D(G) - A(G)$ stands for the *Laplacian matrix* of G. The *Laplacian eigenvalues* of G are the eigenvalues of $L(G)$ and they form $\sigma(G)$, the *Laplacian spectrum* of G.

We consider a multi-agent system with n linear agents $\{1, 2, \dots, n\}$ modeled by a graph G. If x_i denotes the state of the agent i, its dynamics is described by the single integrator

$$\dot{\mathbf{x}}(t) = - \sum_{j \in N(i)} (x_i(t) - x_j(t)),$$

where $N(i)$ denotes the set of neighbours of i. The compact dynamics can be written as $\dot{\mathbf{x}}(t) = -L(G)\mathbf{x}(t)$, where $\mathbf{x}$ is the vector of the agents' states and $L(G)$ is the graph Laplacian.

Following [6] by ℓ and f we denote affiliations with leaders and followers. A *follower graph* G_f of G is the subgraph induced by the set of followers. Consequently, the graph Laplacian $L(G)$ of G may be written as

$$L(G) = \begin{pmatrix} \mathcal{L}_f(G) & l_{f\ell}(G) \\ l_{f\ell}^{\intercal}(G) & \mathcal{L}_\ell(G) \end{pmatrix}. \tag{I.1}$$

The control system we consider is the leader-follower system

$$\begin{pmatrix} \dot{\mathbf{x}}_f(t) \\ \dot{\mathbf{u}}(t) \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_f(G) & l_{f\ell}(G) \\ l_{f\ell}^{\intercal}(G) & \mathcal{L}_\ell(G) \end{pmatrix} \begin{pmatrix} \mathbf{x}_f(t) \\ \mathbf{u}(t) \end{pmatrix},$$

where followers evolve through the Laplacian-based dynamics

$$\dot{\mathbf{x}}_f(t) = -\mathcal{L}_f(G)\mathbf{x}_f(t) - l_{f\ell}(G)\mathbf{u}(t), \tag{I.2}$$

and $\mathbf{u}$ denotes the external control signal ran by the leaders' states.

The system modeled by (I.2) is said to be *controllable* if it can be driven from any initial state to any desired final state in a finite time. In the study of the controllability of multi-agent systems, the main problem is to determine the locations of leaders under which the controllability can be realized. The multi-agent system (I.2) is said to be k*-leaders controllable* if there exist minimum number of k leaders to make (I.2) controllable. In particular, if $k = 1$, the system (I.2) is called *single leader controllable*.

We recall a useful argument for further analysis of controllability of multi-agent systems.

Lemma I.1. ([5]) *The system* (I.2) *is controllable if and only if there is no eigenvector for $L(G)$ taking 0 on all entries corresponding to leaders, i.e. if and only if $L(G)$ and $\mathcal{L}_f(G)$ do not share any common eigenvalues.*

Multi-agent systems arise in many areas of science and engineering (see for example [1], [5], [7], [8], [10], [12]). In this paper we focus on controllability of chain graphs, in particular to chain graphs with one repeated degree. Chain graphs are $2K_2, C_3, C_5$ graphs, which implies that they are also bipartite graphs. We determine the minimum number of leaders needed to make the corresponding system (I.2) modeled by such a graph controllable and provide the locations of leaders in the graph.

The paper is organized as follows. In Section II we give some preliminary results on the structure of chain graphs and on their spectrum. In Section III we present several results concerning Laplacian spectrum and eigenvectors of chain graphs with one repeated degree. In Section IV we consider the controllability of systems (I.2) modeled by a corresponding chain graph. In Section V we present several concluding remarks.

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II. Preliminaries

The Laplacian matrix $L(G)$ of any graph G is symmetric and positive semidefinite. Moreover, 0 is an eigenvalue of G afforded by the all-1 vector $\mathbf{j}$. Therefore, we may assume that the eigenvalues of G (in fact, the roots of the characteristic polynomial $\phi(L(G), x) = \det(xI - L(G))$) are indexed in non-increasing order and given as follows:

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0\,.$$

We denote by $\sigma(G)$ the spectrum of G, i.e. the multiset of its eigenvalues.

The vertex set of a chain graph G consists of two colour classes that are partitioned into h non-empty cells $\bigcup_{i=1}^{h} U_i$ and $\bigcup_{i=1}^{h} V_i$, respectively. All vertices in U_s are joined to all vertices in $\bigcup_{k=1}^{h+1-s} V_k$, for $1 \leq s \leq h$. Therefore, all vertices in U_i (resp. V_j) are *co-neighbours*, i.e. they share the same set of neighbours. If $m_s = |U_s|$ and $n_s = |V_s|$, for $1 \leq s \leq h$, then G is denoted by

$$\mathrm{DNG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)\,.$$

A chain graph is sketched in Figure II.1.

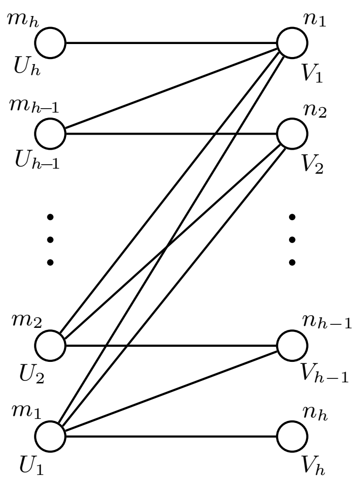

Fig. II.1. The chain graph $G = \mathrm{DNG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

If we follow the vertex ordering from the partition $\left(\bigcup_{i=1}^{h} U_i\right) \cup \left(\bigcup_{i=1}^{h} V_i\right)$, then the *quotient matrix* $Q(G)$ of a chain graph G has the form

$$\left(\begin{array}{cccc|cccc} d_1 & & & & -n_1 & \cdots & -n_{h-1} & -n_h \\ & d_2 & & & -n_1 & \cdots & -n_{h-1} & \\ & & \ddots & & \vdots & \iddots & & \\ & & & d_h & -n_1 & & & \\ \hline -m_1 & \cdots & -m_{h-1} & -m_h & d_1^* & & & \\ -m_1 & \cdots & -m_{h-1} & & & d_2^* & & \\ \vdots & \iddots & & & & & \ddots & \\ -m_1 & & & & & & & d_h^* \end{array}\right). \quad \text{(II.1)}$$

The corresponding diagonal blocks we shortly denote by D_1, D_2, while off-diagonal ones we denote by B_1, B_2.

It is well-known that every eigenvalue of $Q(G)$ is an eigenvalue of G. For more results on spectral properties of chain graphs the reader is referred to [4], [9], [11].

III. Laplacian Spectrum of $\mathrm{DNG}(k, 1, \ldots, 1; 1, \ldots, 1)$

In this section we investigate spectral properties of chain graphs with one repeated degree. These graphs are of the form $\mathrm{DNG}(\underbrace{k, 1, \ldots, 1}_{h}; \underbrace{1, \ldots, 1}_{h})$. Since G has only one repeated degree, then $k > h$.

Theorem III.1. *Let* $\mathrm{DNG}(\underbrace{k, 1, \ldots, 1}_{h}; \underbrace{1, \ldots, 1}_{h})$*,* $k > h$*. Then*

$$\sigma(G) = \{0, h^{k-1}, \kappa_1, \kappa_2, \ldots, \kappa_{2h-1}\},$$

where

$$\begin{cases} \kappa_i \in (i-1, i), & i \in \{1, \ldots, h-1\} \\ \kappa_{h+i} \in (k+i-1, k+i), & i \in \{1, \ldots, h-1\} \\ \kappa_{2h} \geq k+h, & \end{cases}$$

Proof. Taking into account that $d_i = h + 1 - i$, $1 \leq i \leq h$ and $d_j^* = k + h - j$, $1 \leq j \leq h$ and employing [13, Theorem 3.5], we get that the characteristic polynomial $\phi(L(G), x)$ of $L(G)$ is given by

$$x(x-h)^{k-1} \prod_{i=1}^{k+h-1} (x-i) \left(\frac{1}{p_1} + x \sum_{j=2}^{h} \frac{1}{(x - d_{h+2-j})p_j} + \frac{1}{x - d_1} \right).$$

Since $x(x-h)^{k-1}$ is a factor of $\phi(L(G), x)$, the remaining eigenvalues are the roots of the polynomial

$$p(x) = \prod_{i=1}^{k+h-1} (x-i) \left(\frac{1}{p_1} + x \sum_{j=2}^{h} \frac{1}{(x - d_{h+2-j})p_j} + \frac{1}{x - d_1} \right).$$

Then we have:

- $p(0) = (-1)^{k+h}(2h + k - 1)\frac{(k+h-2)!}{h}$;
- $p(1) = (-1)^{k+h+1}(k + h - 3)(k + h - 3)!$;
- $p(\ell) = (-1)^{k+h+\ell} 2\ell(h + 1)(\ell - 1)!(k + h - 2\ell - 2)!\frac{(k+h-\ell-1)!}{(k+h-2\ell)!}$, for $2 \leq \ell \leq h - 1$;
- $p(k) = (-1)^{h+1}\frac{k!}{(k-h+1)(k-h)}(h - 1)!$
- $p(k + \ell) = (-1)^{h-\ell+1} 2(\ell + 1)\frac{(k+\ell)!}{(k+2\ell-h+1)(k+2\ell-h)}(h - \ell - 1)!$, for $1 \leq \ell \leq h - 1$;
- $p(k + h) = -(k + h - 2) \cdot (h - 1)! < 0$.

From the obtained values, we conclude that $p(0), p(1), \ldots, p(h-1)$ alternate in sign. Therefore, for any $i \in \{1, 2, \ldots, h-1\}$, we have $p(t) = 0$, for some $t \in (i-1, i)$. Similar argument holds for $p(k), p(k+1), \ldots, p(k+h-1)$, and consequently, for every $i \in \{k+1, \ldots, k+h-1\}$ we have $p(t) = 0$ for some $t \in (i-1, i)$. Also, since p is a monic polynomial and $p(k+h) < 0$, it follows that $p(t) = 0$ holds for some $t > k + h$. □

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We illustrate the results of Theorem III.1 on the following example.

Example III.2. Let $G = \mathrm{DNG}(6, 1, 1, 1, 1; 1, 1, 1, 1, 1)$. Then

$$\sigma(G) = \{13.03, 9.64, , 08.6, 7.58, 6.58, 3.82, 2.86, 1.92, 0.96\} \cup \{5^5, 0\}.$$

Next we observe the structure of eigenvectors of $L(G)$ corresponding to non-integer eigenvalues.

Theorem III.3. *Let $G = \mathrm{DNG}(k, 1, \dots, 1; 1, 1, \dots, 1)$ with $k > h$, μ a non-integer eigenvalue of G and $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ an associated eigenvector. Then $x_i \neq 0$ for any $1 \le i \le k$.*

Proof. We recall first, (see, for example, [13]) a relation between the eigenvectors of $Q(G)$ and those of G for the same eigenvalue. A vector $\mathbf{v} = (y_1, y_2, \dots, y_h, z_1, z_2, \dots, z_h)^\intercal$ is an eigenvector of $Q(G)$ for μ, if and only if the corresponding eigenvector of G for the same eigenvalue has the form

$$\mathbf{x} = (\underbrace{y_1, y_1, \dots, y_1}_{k}, y_2, \dots, y_h, z_1, \dots, z_h)^\intercal.$$

Assume on the contrary that $\mathbf{x}$ is an eigenvector for the non-integer eigenvalue μ of $L(G)$ such that $x_i = 0, 1 \le i \le k$. By [13, Lemma 3.4], μ is also an eigenvalue of $Q(G)$. So there exists a non-zero vector $(\mathbf{y} \quad \mathbf{z})^\intercal \in \mathbb{R}^{2h}$ such that $Q(G)(\mathbf{y} \quad \mathbf{z})^\intercal = \mu(\mathbf{y} \quad \mathbf{z})^\intercal$ with $y_1 = 0$. Then the eigenvalue equation

$$\begin{pmatrix} D_1 & -B_1 \\ -B_2 & D_2 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \mu \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}$$

can be rewritten as

$$\begin{aligned} D_1\mathbf{y} - B_1\mathbf{z} &= \mu\mathbf{y} \\ -B_2\mathbf{y} + D_2\mathbf{z} &= \mu\mathbf{z}. \end{aligned}$$

The matrices B_1, B_2 have full rank, and therefore are invertible. Next, from

$$\begin{aligned} \mathbf{z} &= B_1^{-1}(D_1 - \mu I_h)\mathbf{y} \\ \mathbf{y} &= B_2^{-1}(D_2 - \mu I_h)\mathbf{z}, \end{aligned}$$

we conclude that

$$\mathbf{y} = B_2^{-1}(D_2 - \mu I_h)B_1^{-1}(D_1 - \mu I_h)\mathbf{y},$$

i.e. $\mathbf{y}$ is an eigenvector of

$$P = B_2^{-1}(D_2 - \mu I_h)B_1^{-1}(D_1 - \mu I_h)$$

for the eigenvalue 1. The latter product is the product of two anti-bidiagonal matrices $B_2^{-1}(D_2 - \mu I_h)$ that is

$$\begin{pmatrix} & & & (k-\mu)/k \\ & & & -(k-\mu) \\ & & \iddots & \iddots \\ & (k+h-2-\mu) & & \\ (k+h-1-\mu) & -(k+h-2-\mu) & & \end{pmatrix}$$

and $B_1^{-1}(D_1 - \mu I_h)$

$$\begin{pmatrix} & & & (1-\mu) \\ & & (2-\mu) & -(1-\mu) \\ & & \iddots & \iddots \\ & (h-1-\mu) & & \\ (h-\mu) & -(h-1-\mu) & & \end{pmatrix},$$

and hence it is a tridiagonal matrix with

$$\begin{aligned} p_{1,1} &= \frac{(h-\mu)(k-\mu)}{k} \\ p_{\ell,\ell} &= (h+1-\ell-\mu)\Big(\frac{k+\ell-1-\mu}{m_{h+1-\ell}} + (k+\ell-2-\mu)\Big), \end{aligned}$$

$2 \le \ell \le h$,

$$\begin{aligned} p_{\ell,\ell-1} &= -(h-\ell+2-\mu)(k+\ell-2-\mu), \quad 2 \le \ell \le h, \\ p_{\ell,\ell+1} &= -\frac{(h-\ell-\mu)(k+\ell-1-\mu)}{m_\ell}, \quad 1 \le \ell \le h-1, \end{aligned}$$

taking into account that $m_1 = k$ and $m_i = 1$, $i \ge 2$. From $\mu \notin \mathbb{Z}$, we have $p_{\ell,\ell-1}, p_{\ell,\ell+1} \neq 0$.

If $y_1 = 0$, then from the first equation in $P\mathbf{y} = \mathbf{y}$ we obtain $y_2 = 0$ ($p_{1,2} \neq 0$). Next, in the similar way, the second equation gives $y_3 = 0$, and so on, until we obtain $y_h = 0$, i.e. $\mathbf{y} = \mathbf{z} = \mathbf{0}$.

Therefore, we obtain that $\mathbf{x} = \mathbf{0}$, which is a contradiction. This completes the proof. □

IV. Controllability of systems modeled by $\mathrm{DNG}(k, 1, \dots, 1; 1, \dots, 1)$

Previously obtained results, in this section will be employed to determine the number of leading agents in (I.2), where the system is modeled by a chain graph $\mathrm{DNG}(k, 1, \dots, 1; 1, \dots, 1)$ for $k > h$.

Theorem IV.1. *Let G be a chain graph $\mathrm{DNG}(k, 1, \dots, 1; 1, \dots, 1)$ with $k > h$. Then the system (I.2) modeled by G is controllable with $k - 1$ co-neighbour vertices in the role of leaders.*

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Proof. The eigenvectors corresponding to the eigenvalue h of the multiplicity $k-1$ are of the form

$$\begin{aligned}
\mathbf{v}_1 &= (\underbrace{1,-1,0,0,\dots,0}_{k},0,\dots,0) \\
\mathbf{v}_2 &= (\underbrace{1,0,-1,0,\dots,0}_{k},0,\dots,0) \\
&\vdots \\
\mathbf{v}_{k-1} &= (\underbrace{1,0,\dots,0,0,-1}_{k},0,\dots,0).
\end{aligned}$$

We first conclude that vertices $\{2,\dots,k\}$ should be selected as leaders. Moreover, any vector corresponding to $\mu = h$ is of the form $(t_1,\dots,t_k,0,\dots,0)^\intercal$. Any $k-1$ of these t_i's cannot be zeros simultaneously. For any $\mathbf{x}_1,\dots,\mathbf{x}_l$, $l < k-1$ there exists $\mathbf{x}_s$, such that $\mathbf{x}_s\mathbf{x}_i = 0$ for each $i, 1 \le i \le l$.

The remaining eigenvalues by Theorem III.1 are non-integer and therefore their eigenvectors, by Theorem III.3 satisfy $x_i \neq 0, 1 \le i \le k$. Now the statement follows by Lemma I.1. □

Example IV.2. For $G = \mathrm{DNG}(6,1,1,1,1;1,1,1,1,1)$ the system (I.2) is 5 leader controllable. The leaders $\ell_1,\dots,\ell_5 \in U_1$ are 5 of 6 vertices with repeated degrees, that are joined to the followers $v_1,\cdots,v_5$ as illustrated in Figure IV.1.

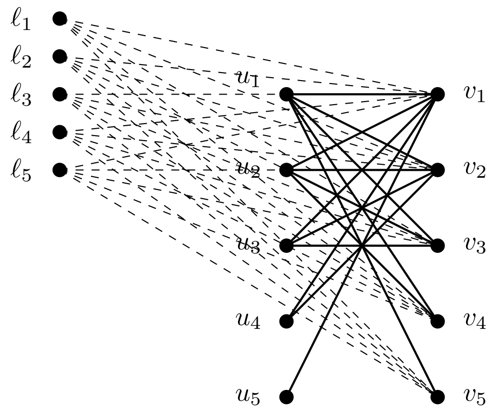

Fig. IV.1. A 5-leader controllable system modelled by $\mathrm{DNG}(6,1,1,1,1;1,1,1,1,1)$.

V. Conclusion

In this paper we have covered the controllability of multi-agent systems that are modelled by special class of bipartite graphs: chain graphs. We have proved that if a chain graph has only one repeated degree with multiplicity k, then the system requires at least $k-1$ controllers in order to be controllable. In this way we positively addressed the questions raised in [7], where the authors asked if there is a family of graphs other than threshold graphs with one multiple degree of multiplicity m for whose controllability at least $m-1$ controllers are needed. Consequently, we expanded the known classes of the controllable multi-agent systems. Taking into account that many engineering systems are modelled by graphs, the obtained results are of particular importance in creating new controllable systems, since the known structures are limited (they mainly include paths, grids, cycles and circulant networks). Another advantageous aspect is a possibility to generate graphs with some desirable properties. One of them is algebraic connectivity, i.e. the second smallest Laplacian eigenvalue. It is a useful tool to measure the robustness and synchronizations of the graphs. For the chain graphs that we considered the algebraic connectivity is always in $(0,1)$ and it approaching to 1 as the size of the graph increases. This brings another benefit, since in general graphs the algebraic connectivity usually decreases if the order of a graph is increased.

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References

[1] T. Kailath, Linear systems, Prentice-Hall, Englewood Cliffs, 1980.
[2] N.V.R. Mahadev, U.N. Peled, it Threshold Graphs and Related Topics, North-Holland, New York, 1995.
[3] K. Ogata, Modern Control Engineering, Prentice-Hall, Upper Saddle River, 2002.
[4] F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, II, Linear Algebra Appl., 429 (2008), 2168–2179.
[5] Z. Ji, Z. Wang, H. Lin, Z. Wang, Interconnection topologies for multi-agent coordination under leader-follower framework, Automatica, (2009), 45(12), 2857–2863.
[6] A. Rahmani, M. Ji, M. Mesbahi, M. Egerstedt, Controllability of multi-agent systems from a graph theoretic perspective, SIAM J. Control Optim., 48 (2009), 162–186.
[7] S-P Hsu, Controllability of the multi-agent system modelled by the threshold graph with one repeated degree, Systems Control Lett., 97 (2016), 149–156.
[8] X. Liu, Z. Ji, Controllability of multiagent systems based on path and cycle graphs, Int. J. Robust Nonlinear Control, 28 (2016), 296–309.
[9] M. Anđelić, S.K. Simić, D. Živković, E. Dolićanin, Fast algorithms for computing the characteristic polynomial of threshold and chain graphs, Appl. Math. Comput. 332 (2018), 329-337.
[10] M. Anđelić, M. Brunetti, Z. Stanić, Laplacian controllability for graphs obtained by some standard products, Graphs Combin., 36 (2020), 1593–1602.
[11] K.Ch. Das, A. Alazemi, M. Anđelić, On energy and Laplacian energy of chain graphs, Discrete Appl. Math., 284 (2020), 391–400.
[12] A. Farrugia, T. Koledin, Z. Stanić, Controllability of NEPSes of graphs, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2020.1778622
[13] A. Alazemi, M. Anđelić, K.Ch. Das, C.M. da Fonseca, Chain graph sequences and Laplacian spectra of chain graphs, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2022.2036672

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