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Research Article

Regular graphs with a cycle as a star complement

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Abstract

Let G be an n-vertex graph having an eigenvalue μ of multiplicity k. A star complement for μ in G is an induced subgraph H with $n-k$ vertices, such that μ is not its eigenvalue. In the case when H is a t-vertex cycle C_t with $t \geq 3$, it is shown that G is regular if and only if $\mu \in \{3, 1, 0, -1, -2\}$. For $\mu = 3$ and $\mu = 1$, G is the complete graph K_4 and the Petersen graph, respectively. For $\mu \in \{0, -1\}$, a structural characterization of infinite families of graphs that appear in the role of G is given, and their existence is shown. The obtained results, together with the result of [F. K. Bell, *Linear Algebra Appl.* **296** (1999) 15–25] concerning $\mu = -2$, establish a complete characterization of regular graphs having C_t as a star complement for some eigenvalue.

Keywords: adjacency matrix; star complement; regular graph; circulant graph; inverse.

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1. Introduction

Let $G = (V, E)$ be a finite undirected graph without loops or multiple edges. The number of vertices of G is called the *order*. By the *eigenvalues* and the *spectrum* of G, we mean the eigenvalues and the spectrum of the standard $\{0, 1\}$-adjacency matrix A_G.

The *neighbourhood* $N(u)$ of a vertex u in G consists of all its neighbours in G. The *closed neighbourhood* is the union $\{u\} \cup N(u)$. Two vertices that share the same neighbourhood (respectively, closed neighbourhood) are called *twins* (*co-twins*), sometimes duplicate (co-duplicate) vertices.

If μ is an eigenvalue of a graph G of multiplicity k, then a *star set* for μ is a set X of k vertices such that μ is not an eigenvalue of the induced subgraph H obtained by removing X. In this situation, H is referred to as a *star complement* for μ in G. Star sets and star complements exist for any eigenvalue of any graph; they need not be unique. The H-neighbourhoods (i.e., the sets of neighbours in H) of vertices in X are non-empty and mutually distinct provided $\mu \notin \{0, -1\}$ [5, Proposition 5.1.4]. Consequently, there are only finitely many graphs with a prescribed star complement for an eigenvalue other than 0 or -1. The eigenvalues 0 and -1 pose a little obstruction to the general theory, as for $\mu \in \{0, -1\}$ and any graph H not having μ as an eigenvalue, there is an infinite family of graphs with H as a star complement for μ. To see this, it is sufficient to observe that adding a twin (respectively, co-twin) vertex increases the multiplicity of the eigenvalue 0 (respectively, -1) by 1. We proceed with more details.

Theorem 1.1 (see [5, Theorem 5.1.7]). *Let X be a set of vertices in a graph G and suppose that G has the adjacency matrix*

$$A_G = \begin{pmatrix} A_X & B^\intercal \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of $G - X$ and

$$\mu I - A_X = B^\intercal (\mu I - C)^{-1} B. \tag{1}$$

This theorem is known as the Reconstruction Theorem. Clearly, the submatrix C is the adjacency matrix of a star complement H. If $|X| = k$ and the order of G is n, then the order of H is $n - k$. In what follows we set $t = n - k$. For $u \in X$, denote by $\mathbf{b}_u$ the vector-column of B corresponding to u. This is the characteristic vector of the H-neighbourhood of u. We define a bilinear form on $\mathbb{R}^t$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\intercal (\mu I - C)^{-1} \mathbf{y}$. Equating the entries in (1), we arrive at

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

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The eigenspace of μ consists of vectors $\begin{pmatrix}\mathbf{x}\\(\mu I-C)^{-1}B\mathbf{x}\end{pmatrix}$, for $\mathbf{x}\in\mathbb{R}^k$. If G is regular and μ is not the largest eigenvalue, then its eigenspace is orthogonal to the all-1 vector, and this yields

$$\langle \mathbf{b}_u, \mathbf{j}\rangle = -1, \tag{3}$$

where $\mathbf{j}$ is the all-1 vector in $\mathbb{R}^t$ [5, Proposition 5.2.4].

There is extensive literature concerning characterizations and constructions of graphs using star complements. They played a crucial role in determining graphs whose spectrum is bounded below by -2 [4]. In particular, regular graphs with prescribed star complements are intensively studied. A survey can be found in [8, 10].

In this paper, we consider regular graphs G with a t-vertex cycle $C_t, t\geq 3$, in the role of a star complement for an eigenvalue μ. We show that this occurs if and only if μ takes one of the following values: $3, 1, 0, -1$ or -2. For $\mu=-2$, we know form [2] (see also [3]) that G is an induced subgraph of the line graph $L(K_t)$. Concerning the remaining possibilities, we determine all regular extensions of C_t for $\mu\in\{3,1\}$, prove the existence, and give a detailed structural characterization of such graphs for $\mu\in\{0,-1\}$. A summary is given in Theorem 2.4.

Our notation is standard. For undefined notions, we refer the reader to any of [5, 10]. In addition, we assume that the reader is familiar with the basic results of the spectral graph theory. In particular, the spectrum of $C_t, t\geq 3$, consists of the numbers $2\cos\frac{2\pi}{t}j$, $1\leq j\leq t$, so it lies in $[-2, 2]$. Thus, 2 is an eigenvalue for every t, 0 is an eigenvalue when $t\not\equiv 2 \pmod 4$, and -1 is an eigenvalue when $t\equiv 0 \pmod 3$. There are similar conclusions for 1 and -2.

The results are reported in the next section.

2. Results

A regular graph of vertex degree r is referred to as an r-regular graph. The canonic orthonormal basis of $\mathbb{R}^t$ is denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_t\}$. The Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x}\cdot\mathbf{y}$.

The first lemma collects results that can be found in some references under different notation or in a different context, see [7, 8, 11]. For the sake of completeness, we include a short proof.

Lemma 2.1. *Let H be a t-vertex r-regular star complement for μ in an s-regular graph G, and suppose that the corresponding star set X counts k vertices. The following hold true:*

(i) Every row sum in $(\mu I - A_H)^{-1}$ is equal to $\frac{1}{\mu-r}$.

(ii) For $\mu\neq s$, every vertex $u\in X$ is adjacent to $r-\mu$ vertices in H; X induces an $(s-r+\mu)$-regular graph.

(iii) If $\mu\neq s$, then $(s-r)t=(r-\mu)k$.

Proof. (i): Since the all-1 eigenvector is associated with the eigenvalue $\mu-r$ of $\mu I-A_H$, the same eigenvector is associated with $\frac{1}{\mu-r}$ in the inverse matrix. This means the every row sum in $(\mu I-A_H)^{-1}$ equals $\frac{1}{\mu-r}$, and we are done.

(ii): Let $u\in X$ be adjacent to p vertices of H. Together with (3), part (i) yields

$$-1=\langle \mathbf{b}_u,\mathbf{j}\rangle = \mathbf{b}_u\cdot\frac{1}{\mu-r}\mathbf{j}=\frac{p}{\mu-r},$$

giving $p=r-\mu$. The second part follows since the degree of u in G is s and exactly $r-\mu$ its neighbours are in H.

(iii) This part follows by counting the number of edges between X and $V(H)$ in two ways. □

Henceforth, we suppose that $H\cong C_t$, for $t\geq 3$, and that its vertices are labelled by $0, 1, \dots, t-1$ in the natural order.

Lemma 2.2. *A cycle C_t is a star complement for $\mu>2$ in a regular graph G if and only if $t=3$ and $G\cong K_4$.*

Proof. Obviously, C_3 is a star complement for $\mu=3$ in K_4.

Suppose that C_t is a star complement for $\mu>2$ in a regular graph G. On the basis of the Interlacing Theorem, we find that μ is a simple eigenvalue, which means that G has $t+1$ vertices. Since G is regular, μ coincides with its vertex degree; necessarily $\mu=3$ (as C_t is 2-regular). Therefore, every vertex of C_t has degree 3 in G, and the same holds for the unique vertex of the star set. Clearly, this occurs only if $G\cong K_4$. □

In the remainder of the paper, we consider the case $\mu\leq 1$. In this case, μ is not the largest eigenvalue of G, and thus the equality (3) and Lemma 2.1(ii)–(iii) hold.

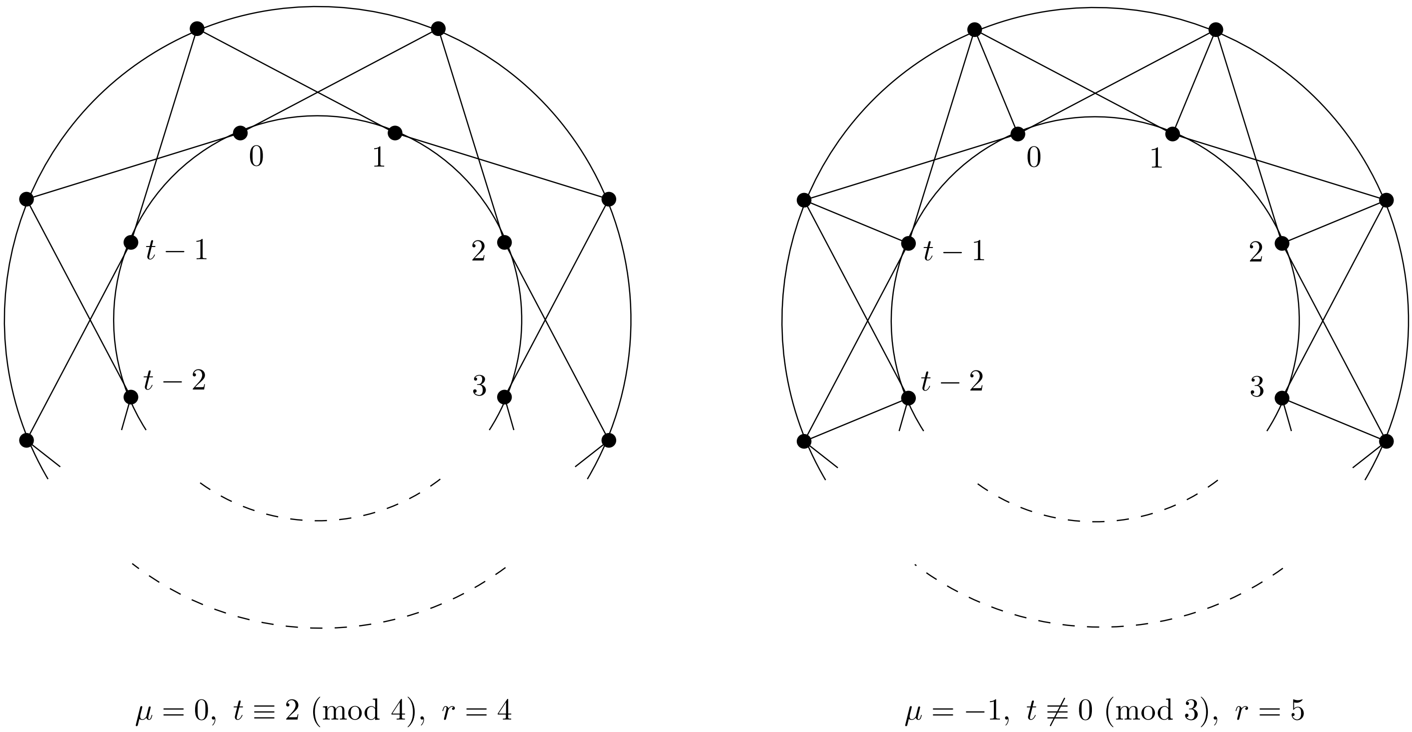

Figure 2.1: Regular graphs with C_t $(t \geq 4)$ as a star complement for 0 or -1.

A *circulant matrix* is a square matrix in which all rows are composed of the same elements and each row is rotated one element to the right relative to the preceding row. We will use the following result:

Lemma 2.3 (Searle [9]). *The inverse of* $(\mu I - A_{C_t})$ *is a circulant matrix with first row* $(s_0, s_1, \dots, s_{t-1})$, *where*

$$s_j = \begin{cases} \dfrac{1}{\sqrt{3}}\big(\sin j\theta + \sin(t-j)\theta\big)(\cos t\theta - 1)^{-1}, & \text{for } \mu \in (-2,\, 2), \\[2ex] \dfrac{1}{\sqrt{\mu^4 - 4}}\Big(\dfrac{z_1^j}{1 - z_1^t} - \dfrac{z_2^j}{1 - z_2^t}\Big), & \text{for } \mu \in (-\infty, -2) \cup (2, \infty), \end{cases}$$

with $\theta = \arccos(\mu/2)$, $z_1 = \frac{\mu - \sqrt{\mu^2 - 4}}{2}$, *and* $z_2 = \frac{\mu + \sqrt{\mu^2-4}}{2}$.

The case $|\mu| = 2$ is covered in given reference, but not in the previous lemma since it is not relevant for this paper. We proceed with $\mu = 1$.

Theorem 2.1. *A cycle* C_t *is a star complement for* 1 *in a regular graph* G *if and only if* $t = 5$ *and* G *is the Petersen graph.*

Proof. Assume that C_t acts as a star complement for 1 in a regular graph G. By Lemma 2.1(ii), every vertex of X is adjacent to exactly one vertex of C_t. From $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = 1$, we deduce that the first entry of the first row of $(I - A_{C_t})^{-1}$ (denoted by s_0 in Lemma 2.3) must be 1. By Lemma 2.3, this entry is $\frac{1}{\sqrt{3}} \sin \frac{t\pi}{3} (\cos \frac{t\pi}{3} - 1)^{-1}$, which yields $t \equiv 5 \pmod 6$.

For $t = 5$, G has 10 vertices and its degree is 3 (Lemma 2.1(iii)). A simple structural consideration based on (2) leads to the Petersen graph.

For $t \geq 11$, we deduce that the 6th entry of the first row of $(I - A_{C_t})^{-1}$ is

$$\frac{1}{\sqrt{3}}\Big(\sin\frac{5\pi}{3} + \sin\frac{(t-5)\pi}{3}\Big)\Big(\cos\frac{t\pi}{3} - 1\Big)^{-1} = 1,$$

since $t - 5$ is divisible by 6. If $u, v \in X$ are vertices adjacent to the vertices labelled by 0 and 5 in C_t, respectively, then $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 1$, which contravenes the equality (2). This denies the existence of G, and we are done.

Conversely, the fact that the Petersen graph contains C_5 as a star complement for 1 is checked directly. □

A referee remarked that the result of the previous theorem is obtained in [1] in a slightly different context. The same reference offers examples of regular graphs with a cycle as a star complement for 0 or -1. We continue with particular families of graphs with C_t as a star complement for $\mu \in \{0, -1\}$.

Lemma 2.4. *The graphs of Figure 2.1 have C_t as a star complement for 0 and -1, respectively.*

Proof. Since both graphs contain C_t as an induced subgraph, it is sufficient to prove that this cycle features a star complement. Since $t \equiv 2 \pmod 4$ for the first graph and $t \not\equiv 0 \pmod 3$ for the second one, the eigenvalue in question does not belong to the spectrum of C_t. Thus, it remains to show that (2) holds for every vertex in X.

Let G be the first graph. For every $u \in X$, we have $\langle \mathbf{b}_u, \mathbf{b}_u\rangle = 0$ since $C_t + u$ is a bipartite graph of an odd order. Due to Lemma 2.3, the first row of $-A_{C_t}^{-1}$ is $(0, -\frac{1}{2}, 0, \frac{1}{2}, 0, -\frac{1}{2}, \dots, 0, -\frac{1}{2})$. For $u, v \in X$ and their C_t-neighbourhoods $N_C(u) = \{i-1, i+1\} \neq \{j-1, j+1\} = N_C(v)$ (where the vertices of C_t are taken $\bmod\ t$), we compute

$$\langle \mathbf{b}_u, \mathbf{b}_v\rangle = -\mathbf{b}_u A_{C_t}^{-1}\mathbf{b}_v = -\mathbf{e}_i \cdot \mathbf{b}_v = \begin{cases} -1, & \text{if } i \text{ and } j \text{ are consecutive vertices in } C_t,\\ 0, & \text{otherwise,}\end{cases}$$

as desired.

If G is the second graph of Figure 2.1, then the first row of $(-I - A_{C_t})^{-1}$ follows different patterns depending on t. For $t \equiv 1 \pmod 3$, it reads $\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \dots, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$; it starts with $-\frac{1}{3}$, and then the triple $-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}$ is repeated $\frac{t-1}{3}$ times. For $t \equiv 2 \pmod 3$, it reads $\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \dots, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$; the previous triple is negated and repeated $\frac{t-2}{3}$ times, and the entries $\frac{1}{3}, -\frac{2}{3}$ are inserted at the end.

For $N(u) = \{i-1, i, i+1\}$, $\langle \mathbf{b}_u, \mathbf{b}_u\rangle$ is the entry sum of

$$\frac{1}{3}\begin{pmatrix} -1 & -1 & 2\\ -1 & -1 & -1\\ 2 & -1 & -1\end{pmatrix}, \text{ for } t \equiv 1 \pmod 3, \quad \text{and} \quad \frac{1}{3}\begin{pmatrix} 1 & -2 & 1\\ -2 & 1 & -2\\ 1 & -2 & 1\end{pmatrix}, \text{ for } t \equiv 2 \pmod 3.$$

Hence, it is -1 in both cases.

For $N(v) = \{j-1, j, j+1\}, j \neq i$, and any admissible t, we have

$$\langle \mathbf{b}_u, \mathbf{b}_v\rangle = -\mathbf{b}_u(I + A_{C_t})^{-1}\mathbf{b}_v = -\mathbf{e}_i \cdot \mathbf{b}_v,$$

along with the conclusion as in the previous case. This completes the proof. □

One may observe that the previous graphs have been constructed by employing Lemma 2.1(ii)–(iii). Part (ii) tells us that every vertex outside C_t has exactly two (respectively, three) neighbours in C_t for the first (second) graph. Part (iii) gives the order and the degree of a constructed graph. The first graph is recognized in the literature as a 'cyclotomic toral tessellation' (cf. [6]); it is also recognized in the theory of signed graphs, as it underlies signed graphs with exactly two eigenvalues: 2 and -2 [6, 12]. We proceed with a characterization for $\mu \in \{0, -1\}$.

Theorem 2.2. *A cycle C_t is a star complement for $\mu \in \{0, -1\}$ in an r-regular graph if and only if either (a) $\mu = 0$, $t \equiv 2 \pmod 4$ and r is even or (b) $\mu = -1$, $t \not\equiv 0 \pmod 3$ and $r \equiv 2 \pmod 3$.*

Proof. Assume first that C_t is a star complement for the required eigenvalue in an r-regular graph G. Since μ does not appear in the spectrum of C_t, we have $t \equiv 2 \pmod 4$ for $\mu = 0$, and $t \not\equiv 0 \pmod 3$ for $\mu = -1$. From Lemma 2.1(iii), we have $(r-2)t = (2-\mu)|X|$, giving $r = 2(|X| + t)$ for $\mu = 0$, which completes (a). For $\mu = -1$, we obtain $r = 3|X| + 2t \equiv 2 \pmod 3$, as t is not divisible by 3, which completes (b).

Suppose now that (a) holds. We prove the existence of a desired r-regular graph G by constructing it. For $r = 4$, on the basis of Lemma 2.4, we may take G to be the first graph of Figure 2.1. For $r \geq 6$, G is obtained from the previous graph by inserting $\frac{r-4}{2}$ twins to every vertex outside C_t. Indeed, the multiplicity of 0 is increased by $\frac{r-4}{2}$ (as the addition of a twin increases its multiplicity by 1), to equals $|V(G)| - t$, as desired. Moreover, G is regular because every vertex of the initial graph has received $r-4$ new neighbours, whereas every additional vertex has a twin, necessarily of the same degree.

For (b), we act in a very similar manner. For $r = 5$, the second graph of Figure 2.1 will do. For $r \geq 8$, the desired graph is obtained by inserting $\frac{r-5}{3}$ co-twins to every vertex outside C_t. The multiplicity of -1 and the regularity of G are considered as before. □

Lemma 2.1(iii) implies that for (a) G has $\frac{rt}{2}$ vertices, and for (b) it has $\frac{(r+1)t}{3}$ vertices.

The row sum of $(\mu I - A_{C_t})^{-1}$ is computed in Lemma 2.1(i). At this point, we need the following lemma which establishes the sum of even and the sum of odd entries in each row, when t is even.

Lemma 2.5. *For t even, let $(s_0, s_1, \dots, s_{t-1})$ be the first row of $(\mu I - A_{C_t})^{-1}$. Then*

$$\sum_{j=0}^{\frac{t-2}{2}} s_{2j} = -\frac{\mu}{(2-\mu)(2+\mu)} \quad \textit{and} \quad \sum_{j=0}^{\frac{t-2}{2}} s_{2j+1} = -\frac{2}{(2-\mu)(2+\mu)}.$$

Proof. Denote the first sum of the statement formulation by Σ_{even}, and the second one by Σ_{odd}. Considering the identity $(\mu I - A_t)(\mu I - A_t)^{-1} = I$ and taking into account that $(\mu I - A_t)^{-1}$ is a circulant matrix, we display the system

$$\mu s_0 - s_{t-1} - s_1 = 1,\ \mu s_{2j} - s_{2j-1} - s_{2j+1} = 0,\ 1 \le j \le \frac{t-2}{2}.$$

Summing up these equalities, we arrive at $\mu\Sigma_{\text{even}} - 2\Sigma_{\text{odd}} = 1$. From Lemma 2.1(i), we have $\Sigma_{\text{even}} + \Sigma_{\text{odd}} = \frac{1}{\mu-2}$. Solving the last system, we arrive at the desired results. □

In the final step, we eliminate the possibility $\mu \le -3$.

Theorem 2.3. *There is no regular graph containing C_t as a star complement for $\mu \le -3$.*

Proof. For a contradiction, suppose that G is an r-regular graph satisfying the statement assumptions. By Lemma 2.1(ii), μ is an integer and every vertex in X is adjacent to $2-\mu$ vertices in C_t. As before, let $\mathrm{s} = (s_0, s_1, \dots, s_{t-1})$ denote the first row of $(\mu I - A_{C_t})^{-1}$. From Lemma 2.3, we deduce

$$s_j = \frac{1}{\sqrt{\mu^4-4}} \frac{z_1^j(1-z_2^t) - z_2^j(1-z_1^t)}{(1-z_1^t)(1-z_2^t)} = \frac{1}{\sqrt{\mu^4-4}} \frac{(z_1^j - z_2^j + z_1^{t-j} - z_2^{t-j})}{(1-z_1^t)(1-z_2^t)},$$

where the last equality follows since $z_1z_2 = 1$ (see the same lemma).

To simplify the previous expression, we observe that z_1 and z_2 are the solutions of $x^2 - \mu x + 1 = 0$, which is the characteristic equation for the recurrence

$$\begin{cases} F_{j+2}(\mu) = \mu F_{j+1}(\mu) - F_j(\mu),\ j \ge 2, \\ F_0(\mu) = 0, \\ F_1(\mu) = -\sqrt{\mu^2-4}, \end{cases} \tag{4}$$

defining the corresponding Fibonacci polynomials. Solving the recurrence, we arrive at $F_j(\mu) = z_1^j - z_2^j$, giving

$$s_j = \frac{1}{\sqrt{\mu^4-4}} \frac{F_j(\mu) + F_{t-j}(\mu)}{(1-z_1^t)(1-z_2^t)}, \tag{5}$$

Assume that t is even. We claim that, for $0 \le j \le \frac{t-2}{2}$, s_{2j} is negative and s_{2j+1} is positive. Since $z_1 < -1 < z_2 < 0$, we have $(1-z_1^t)(1-z_2^t) < 0$. Concerning the signature, we obtain

$$\operatorname{sign}(s_j) = -\operatorname{sign}\big(F_j(\mu) + F_{t-j}(\mu)\big). \tag{6}$$

Observing once again the recurrence, we easily deduce that for $i \ge 0$ we have $F_{2i} \ge 0$ with the strict inequality whenever $i > 0$, and $F_{2i+1} < 0$. This proves our claim.

Now, for $u \in X$, $\langle \mathbf{b}_u, \mathbf{b}_u \rangle$ is the entry sum in the submatrix of $(\mu I - A_{C_t})^{-1}$ induced by C_t-neighbours of u. Since there are exactly $2-\mu$ neighbours, by taking into account that the corresponding inverse is circulant, we obtain

$$\mu = \langle \mathbf{b}_u, \mathbf{b}_u \rangle \le (2-\mu) \sum_{s_j<0} s_j = (2-\mu) \sum_{j=0}^{\frac{t-2}{2}} s_{2j} = -\frac{\mu}{2+\mu}, \tag{7}$$

where the last but one (respectively, last) equality follows from the previous claim (Lemma 2.5). Hence, $\mu \ge -3$. This eliminates all the possibilities, except $\mu = -3$. In this case, we have equality in (7), and u is adjacent to exactly 5 vertices of C_t. This means that there are exactly 5 negative entries in s, thus we have $t = 10$, and $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = -3$ holds if and only if the C_t-neighbourhood of u does not contain mutually adjacent vertices. Therefore, $|X| = 2$ and for distinct $u, v \in X$ we compute $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 2$. However, $\langle \mathbf{b}_u, \mathbf{b}_v \rangle$ must be zero since we have $u \nsim v$ because G is regular. This contradiction concludes the proof for t even. (One may avoid computation of the bilinear form, and arrive at a contradiction by computing the least eigenvalue of the corresponding 12-vertex graph.)

Assume now that t is odd. From Lemma 2.1(ii), we deduce that this occurs only when μ is odd, and $t = (2-\mu)i$, for i odd. For $i = 1$, G is a regular cone over C_t, which occurs only when $t = 3$, but then $G \cong K_4$, a contradiction.

For $i \ge 3$, we follow the idea of the previous part of the proof, that is we show that the sum of the $2-\mu$ smallest entries of s is greater than $\frac{\mu}{2-\mu}$, as this would imply $\langle \mathbf{b}_u, \mathbf{b}_u \rangle > \mu$, for $u \in X$.

We have $(1-z_1^t)(1-z_2^t)>0$, which together with the recurrence (4) leads to $\operatorname{sign}(s_j)=\operatorname{sign}(F_j(\mu)+F_{t-j}(\mu))$. From (2), we deduce that $s_j=s_{t-j}$, for $1\leq j\leq t-1$. Thus, negative entries of s are s_0 and pairs s_{2j},s_{t-2j}, for $2\leq 2j<\frac{t}{2}$. In addition, again from (2), we have $s_{2j}<s_{2j+2}$, for $2j+2<\frac{t}{2}$. In other words, the $2-\mu$ smallest entries of s are s_0 and pairs $s_2,s_{t-2},s_4,s_{t-4},\dots,s_{1-\mu},s_{t-1+\mu}$. Let $\mathrm{s}'=(s'_0,s'_1,\dots,s'_{t-3+\mu})$ be the first row of $(\mu I-A_{t-2+\mu})^{-1}$. By employing (5) and using

$$z_1^j+z_2^j=\frac{1}{\sqrt{\mu^4-4}}(F_{j-1}(\mu)-F_{j+1}(\mu)),$$

we compute $\operatorname{sign}(s_j-s'_j)=\operatorname{sign}(z_1^j+z_2^j)$, which is 1 for j even, and this yields $s_{2j}>s'_{2j}$. Since $s'_{2j}=s'_{t-2j}$, for $2j\leq\frac{t}{2}$, the previous inequality yields the following relations ($\vee$ is the inequality symbol):

$$\begin{array}{c|c|c|c|c|c|c|c|c|c}
s_0 & s_1 & s_2 & \cdots & s_{1-\mu} & \cdots & s_{t-1+\mu}=s_{1-\mu} & \cdots & s_{t-2}=s_2 & s_{t-1}\\
\vee & & \vee & & \vee & & \vee & & \vee & \\
s'_0 & s'_1 & s'_2 & \cdots & s'_{1-\mu} & \cdots & s'_{t-1+\mu}=s'_{1-\mu} & \cdots & s'_{t-1+\mu}=s'_2 & s'_{t-3+\mu}
\end{array}$$

Now, we compute

$$s_0+\sum_{j=1}^{\frac{1-\mu}{2}}(s_{2j}+s_{t-2j})>s'_0+\sum_{j=1}^{\frac{1-\mu}{2}}(s'_{2j}+s'_{t-2+\mu-2j})\geq-\frac{\mu}{(2-\mu)(2+\mu)}\geq\frac{\mu}{2-\mu},$$

for $\mu\leq-3$, where the second inequality is a consequence of (7). This completes the entire proof. □

In light of the previous result, one may ask whether there are non-regular graphs containing C_t as a star complement for an integer $\mu\leq-3$. There are such graphs, and here are some examples:

- If G is a cone over C_t, then its least eigenvalue is $1-\sqrt{t+1}$ (see [5, Theorem 2.1.8]). Thus, if $t+1$ is a square, G is a one-vertex extension of C_t for $\mu=1-\sqrt{t+1}$.
- Another one-vertex extension has been met in the proof of Theorem 2.3. This is an 11-vertex graph containing C_{10} as a star complement for -3.
- We have employed the supporting software SCL – Star Complement Library [13] to verify that for $t\leq 20$, apart from the aforementioned graphs, there is only one example: The 21-vertex graph containing C_{15} as a star complement for -3. The set X induces the complete graph K_6, and each of its vertices is adjacent to 10 vertices of C_{15}; the spectrum is $10,2^4,1^5,(-1)^5,(-3)^6$.

Gathering the previous results and bearing in mind the result of [2], we arrive at the following conclusion.

Theorem 2.4. *Let G be an r-regular graph containing C_t as a star complement for an eigenvalue μ. Then one of the following holds:*

(i) $\mu=3$, $t=3$, *and* $G\cong K_4$.

(ii) $\mu=1$, $t=5$, *and G is the Petersen graph.*

(iii) $\mu=0$, $t\equiv 2\pmod 4$, *r is even, and G belongs to an infinite family and has $\frac{rt}{2}$ vertices.*

(iv) $\mu=-1$, $t\not\equiv 0\pmod 3$, $r\equiv 2\pmod 3$, *and G belongs to an infinite family and has $\frac{(r+1)t}{3}$ vertices.*

(v) (see [2]) $\mu=-2$, *t is odd, and G is a an induced subgraph of the line graph $L(K_t)$.*

Proof. Lemma 2.2 considers the case $\mu\geq 3$ and establishes part (i) of the statement formulation. We have $\mu\neq 2$ since 2 is an eigenvalue of C_t, whereas Theorem 2.3 ensures that $\mu>-3$. In addition, due to Lemma 2.1(ii), μ is an integer. Parts (ii), (iii), and (iv) follow from Theorems 2.1 and 2.2. The last part follows from [2]. □

Theorem 2.4 is a refinement of [2, Theorem 2.4], where the case $t=3$ is omitted. We already pointed out that part (ii) can be found in [1].

If G is an n-vertex r-regular graph, then λ ($\neq n-r-1$) is an eigenvalue of its complement $\overline{G}$ if and only if $-\lambda-1$ is an eigenvalue of G (see any of [5, 10]). Together with the previous theorem, this yields the following conclusion: If G is an n-vertex $(n-r-1)$-regular graph containing $\overline{C}_t$ (the complete graph K_t with a Hamiltonian cycle removed) as a star complement for an eigenvalue μ ($\neq n-r-1$), then $\mu\in\{-2,-1,0,1\}$ and G is the complement of a graph of Theorem 2.4(ii)–(v). For $\mu=n-r-1$, G cannot be connected, since it would have $t+1$ vertices of the same degree, which is impossible as $\overline{C}_t$ has t vertices of degree $t-3$. If G is disconnected, then either $t=3$ (with $\mu=1$ and $G\cong 3K_2$) or $t=4$ (with $\mu=2$ and $G\cong 2K_3$).

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