ROW SPACE CARDINALITIES

MIODRAG ŽIVKOVIĆ

ABSTRACT. Let $\mathcal{B}_n$ be the set of all $n \times n$ Boolean matrices. Let $\mathrm{R}(A)$ denote the row space of $A \in \mathcal{B}_n$, let $\mathcal{R}_n = \{r \mid r = \mathrm{r}(A),\ A \in \mathcal{B}_n\}$, and let $a_n = \min\{q \geq 1 \mid q \notin \mathcal{R}_n\}$. By extensive computation we found that

$$\begin{aligned}\mathcal{R}_9 \cap [1, 256] \quad = \quad & [1, 190] \cup [192, 204] \cup \{206\} \cup [208, 212] \cup \{214, 216, 220\} \cup \\ & [224, 228] \cup \{230, 232, 236, 240, 248, 256\},\end{aligned}$$

and therefore $a_9 = 191$. Furthermore, $a_n \geq 5\ \sqrt[11]{336}^{\,n}$ for $n \geq 31$. We proved that if $n \geq 7$, then the set $\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}]$ contains at least

$$n^2 - 7n + 14 + \frac{1}{24}\left((n-8)(n-10)(2n-15) + 3(n \bmod 2)\right)$$

elements.

1. Introduction

Let $\mathcal{B}_{mn}$ denote the set of all $m \times n$ Boolean matrices, and let $\mathcal{B}_n = \mathcal{B}_{nn}$. The set $\mathcal{B}_n$ with the ordinary matrix multiplication and Boolean operations on entries is a semigroup. Let $\mathrm{R}(A)$ denote the row space of A, i.e. the subspace spanned by the rows of A. Analogously, let $\mathrm{C}(A)$ denote the column space of A; then $|\,\mathrm{C}(A)| = |\,\mathrm{R}(A)|$ [1].

Denote $\mathcal{R}_n = \{r \mid r = |\,\mathrm{r}(A)|,\ A \in \mathcal{B}_n\}$. Obviously, $\mathcal{R}_n \subseteq [1, 2^n]$. Konieczny [4] proved that $\mathcal{R}_n \cap (2^{n-1}, 2^n] = \{2^{n-1} + 2^k \mid 0 \leq k \leq n-1\}$, and conjectured that $[1, 2^{n-1}] \subset \mathcal{R}_n$. Li and Zhang [6] proved that Konieczny's conjecture is not true, because if $n > 6$, then $2^{n-1} - 1 \notin \mathcal{R}_n$. Furthermore Hong [3] proved that

$$\mathcal{R}_n \cap \ \left((2^{n-1} - 2^{n-5}, 2^{n-1} - 2^{n-6}) \cup (2^{n-1} - 2^{n-6}, 2^{n-1})\right) = \emptyset, \quad n \geq 7$$

i.e. that there are at least two gap intervals in $\mathcal{R}_n^0 = \mathcal{R}_n \cap [1, 2^{n-1}]$. He also proved that $2^{n-1} - 2^{n-5} \in \mathcal{R}_n$ and $2^{n-1} - 2^{n-6} \in \mathcal{R}_n$.

Breen [2] verified $\mathcal{R}_7$ ($\mathcal{R}_7^0 = [1, 64] \setminus \{61, 63\}$) and obtained $\mathcal{R}_8$:

$$\mathcal{R}_8^0 = [1, 128] \setminus \{109, 111, 117, 119, 121, 122, 123, 125, 126, 127\}.$$

Let $a_n = \min\{q \geq 1 \mid q \notin \mathcal{R}_n\}$. The first 8 members of this sequence are 3, 5, 7, 11, 19, 35, 61 and 109. Zhong [5] proved that $a_n \geq 6\sqrt{2}^{\,n} - 7$ for $n \geq 13$ odd, $a_n \geq \sqrt{32}\sqrt{2}^{\,n} - 7$ for $n \geq 14$ even, and so $a_n \geq \sqrt{32}\sqrt{2}^{\,n} - 7$ for $n \geq 14$.

By extensive computation we obtained the set $\mathcal{R}_9$ and $a_9 = 191$. Using a special construction connecting elements of subsequent sets $\mathcal{R}_n$, we improved the lower bound for a_n: $a_n \geq 5\ \sqrt[11]{336}^{\,n}$ for $n \geq 31$. In the set $\mathcal{R}_n \cap \left(2^{n-2} + 2^{n-3}, 2^{n-1}\right]$,

---

1991 *Mathematics Subject Classification.* 15A36, 15-04.

Key words and phrases. $(0,1)$ matrices, cardinality of row space, permutation equivalent.

Supported by Ministry of Science and Technology of Serbia, Grant 144030.

$n \geq 7$, we found at least

$$n^2 - 7n + 14 + \frac{1}{24}\left((n-8)(n-10)(2n-15) + 3(n \bmod 2)\right)$$

different elements. Because of the agreement with the result of Hong [3], we hypothesize that this set contains no other elements.

Notation. Depending on context, 0, 1 denote numbers or matrices with all elements equal to 0 and 1 respectively; 0_n, 1_n, I_n denote 0-, 1-, and identity matrices in $\mathcal{B}_n$, respectively; $0_{m\times n}$, $1_{m\times n} \in \mathcal{B}_{mn}$ denote 0-, 1-, $m \times n$ matrices, respectively; $A_{i.}$ denotes the ith row of the matrix A. $W(A)$ denotes the weight (the number of ones) in A, and $\mathrm{r}(A) = |\,\mathrm{R}(A)|$ is the row space cardinality of A.

2. The set $\mathcal{R}_9$

We say that matrices A and B from $\mathcal{B}_n$ are permutationaly equivalent, $A \sim B$, if $B = PAQ$, where P, Q are are permutation matrices. Obviously, if $A \sim B$ then $\mathrm{r}(A) = \mathrm{r}(B)$. We obtained $\mathcal{R}_9$ using the list of permutationaly nonequivalent matrices in $\mathcal{B}_8$ [8].

Let A_π denote the lexicographically smallest matrix in the equivalence class containing A; we call it the π-representative of A. Let $\mathcal{B}_n^\pi$ denote the set of π-representatives in $\mathcal{B}_n$. For an arbitrary $B \in B_{n-1}$, let $\mathrm{bord}(B)$ denote the subset of $\mathcal{B}_n$, containing matrices with the upper left minor equal to B. We say that the matrices in $\mathrm{bord}(B)$ are *obtained by extending* B; if $A \in \mathrm{bord}(B)$, then A is an *extension* of B. Furthermore, let $\mathrm{bord}_\pi(B) = \{A_\pi \mid A \in \mathrm{bord}(B)\}$. Williamson [9] noted that if B and B' are equivalent, then $\mathrm{bord}_\pi(B) = \mathrm{bord}_\pi(B')$. Therefore,

$$\mathcal{R}_9 = \cup_{B \in \mathcal{B}_8^\pi} \{\mathrm{r}(A) \mid A \in \mathrm{bord}_\pi(B)\}.$$

Theorem 2.1.

$$\begin{aligned} \mathcal{R}_9^0 \quad = \quad & [1,190] \cup [192,204] \cup \{206\} \cup [208,212] \cup \{214,216,220\} \cup \\ & [224,228] \cup \{230,232,236,240,248,256\} \end{aligned} \tag{2.1}$$

and $a_9 = 191$.

Proof. Denote by $\mathcal{R}$ the set from the right hand side of (2.1). If B is obtained by extending $A \in \mathcal{B}_n$ with zero row and zero column, then $\mathrm{r}(B) = \mathrm{r}(A)$. Therefore, $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$, and $[1,108] \subset \mathcal{R}_9$.

Let $\mathcal{R}(A) = \{\mathrm{r}(B) \mid B \in \mathrm{bord}(A)\}$. After determining $\mathcal{R}(A_i)$, where

$$A_1 = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&1&1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&1&1&1&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \end{bmatrix}.$$

it can be verified that

$[109, 177] \cup \{183\} \subset \mathcal{R}(A_3)$, $[178, 190] \setminus \{183\} \subset \mathcal{R}(A_2)$, and

$$[192, 204] \cup \{206\} \cup [208, 212] \cup \{214, 216, 220\} \cup$$
$$[224, 228] \cup \{230, 232, 236, 240, 248, 256\} \subset \mathcal{R}(A_1),$$

proving that $\mathcal{R} \subset \mathcal{R}_9^0$.

The proof of $\mathcal{R} \subset \mathcal{R}_9^0$ is more complicated, because the row space cardinalities of all extensions of all 14685630688 matrices from $\mathcal{B}_8^\pi$ have to be checked. The actual computation of all these (approximately 2×10^{15}) RSCs is, of course, practically impossible. In order to skip some RSC computations, we used the upper bound on $\mathrm{r}(A)$ determined using only two or three rows of A with the appropriately chosen indices i, j, k (see for example [3]):

$$b_2(A, i, j) = 2^{n-2} + 2^{n-W(A_{i.})} + 2^{n-W(A_{j.})} + 2^{n-W(A_{i.}+A_{j.})},$$

$$\begin{aligned} b_3(A, i, j, k) &= 2^{n-3} + 2^{n-W(A_{i.})} + 2^{n-W(A_{j.})} + 2^{n-W(A_{k.})} + 2^{n-W(A_{i.}+A_{j.})} + \\ &+ 2^{n-W(A_{i.}+A_{k.})} + 2^{n-W(A_{j.}+A_{k.})} + 2^{n-W(A_{i.}+A_{j.}+A_{k.})}, \end{aligned}$$

and

$$b(A, i, j, k) = \min\{b_3(A, i, j, k), b_2(A, i, j)\}.$$

If, for example, $B \in \mathcal{B}_9$ has two rows with at least 5 ones, i.e. $W(B_{i.}) \geq 5$ and $W(B_{j.}) \geq 5$ for some i, j, then $\mathrm{r}(B) \leq b_2(B, i, j) \leq 2^{9-2} + 3 \cdot 2^{9-5} = 176$.

If we already know matrices with RSCs 1, 2, …, g, and we estimate that the upper bound for the next extension B of the current matrix $A \in \mathcal{B}_8^\pi$ is less than or equal to g, then the computation of $r(B)$ can be skipped. Even more, we do not determine $\mathrm{r}(A)$ if the upper bound for $\mathrm{r}(A)$ is less than $g/4$. In our case $g = 190$. This fact is incorporated in Algorithm 2.2, which for given $A \in \mathcal{B}_n$ and g determines $\mathcal{R}(A) \cap (g, 2^n]$. After determining $\mathcal{R}(A) \cap (g, 2^n)$ for all $A \in \mathcal{B}_8^\pi$ by Algorithm 2.2, we obtained that $\mathcal{R}_9^0 \subset \mathcal{R}$, ending the proof. $\square$

Algorithm 2.2. Determine $\mathcal{R}(A) \setminus [1, g]$.

Input : n, g — integers, $A \in \mathcal{B}_n$;
Output : $S = \mathcal{R}(A) \cap (g, 2^n]$.
  $S = \emptyset$;
  $g' = \min\{b(A, 1, 2, 3), b(A^T, 1, 2, 3)\}$;
  if $g' > g/4$ **then**
    $r = \mathrm{r}(A)$;
    if $r > g/4$ **then**
      for $a \in \{0, 1\}^n$ **do**
        $A' = \left[\begin{array}{c} A \\ \hline a \end{array}\right]$;
        $g'' = b(A', 1, 2, n+1)$;
        if $g'' > g/2$ **then**
          for $b \in \{0, 1\}^n$, $c \in \{0, 1\}$ **do**
            $A'' = \left[\begin{array}{c|c} A & b \\ \hline a & c \end{array}\right]$;
            $g''' = b((A'')^T, 1, 2, n+1)$;

            if $g''' > g$ **then**

              $S = S \cup \mathrm{r}(A'')$;

Note the interesting fact that $191 = 2^7 + 2^6 - 1 \notin \mathcal{R}_9$, even though $[1, 2^{n-2} + 2^{n-3}] \subset \mathcal{R}_n$ is true for all $n \leq 8$.

In Table 1 some matrices from $\mathcal{B}_9$ and their RSCs are shown. The compact representation is used: each row is represented by a hexadecimal integer. For example, the entry 189 : [1 2 C 14 24 44 84 109 112] represents the equality

$$\mathrm{r}\begin{bmatrix} 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0&0&1 \\ 1&0&0&0&1&0&0&1&0 \end{bmatrix} = 189.$$

3. A method to obtain lower bounds for a_n

There are cases when the RSC of a matrix can be expressed in terms of the RSC of some its submatrix. For example, if $A \neq 0$ and $\mathrm{r}(A) = a$, then

$$\mathrm{r}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = a, \quad \mathrm{r}\begin{bmatrix} A & 0 \\ 1 & 1 \end{bmatrix} = a+1, \quad \mathrm{r}\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = 2a,$$

TABLE 1. Some matrices in $\mathcal{B}_9$ with interesting RSC values.

r(A)	A								
127	1	2	5	9	14	24	44	84	112
139	1	3	6	A	12	22	42	9C	182
143	1	2	5	9	30	50	90	114	12A
149	1	2	4	9	16	28	48	88	118
151	1	2	5	9	30	50	90	111	126
155	1	2	5	C	14	24	46	C0	140
157	1	3	6	C	30	50	68	A0	120
163	1	3	6	A	12	22	42	84	182
167	1	2	4	9	11	2E	60	C0	140
169	1	3	C	14	24	46	86	140	184
173	1	3	6	18	28	34	50	90	110
175	3	5	9	11	22	60	A0	140	181
179	3	5	9	11	21	42	82	141	181
181	3	5	9	11	21	42	C0	140	181
183	1	6	A	12	22	42	84	109	180
185	1	2	4	18	28	48	90	110	1EF
187	3	5	9	11	21	41	81	106	118
189	1	2	C	14	24	44	84	109	112
197	1	2	4	8	10	21	7E	C0	140
199	1	2	C	14	24	44	84	108	1F3
201	1	2	C	14	24	44	84	10B	114
203	1	2	4	18	28	48	88	110	1E7
209	3	5	9	11	21	41	81	102	10D
211	3	5	9	11	21	41	81	102	10C
225	3	5	9	11	21	41	81	102	105
227	1	2	4	8	10	60	A0	140	19F

Combining these simple rules, we obtain that if $A \neq 0$, then

$$\text{(3.1)}\quad \mathrm{r}\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2a, \qquad \mathrm{r}\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2a+1, \qquad \mathrm{r}\begin{bmatrix} A & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2a+2.$$

Consequently, if there is a matrix $A \in \mathcal{B}_{n-2}$, satisfying $\mathrm{r}(A) = a$, then there are matrices in B_n with the RSCs $2a$, $2a+1$, and $2a+2$. This simple construction enables to obtain a lower bound for a_n, that is by a constant factor sharper than that from [5].

Theorem 3.1. *If $n \geq 9$ then $a_n \geq 6\sqrt{2}\sqrt{2}^{\,n} - 1$.*

Proof. The matrices with RSCs 1, 2 and 3 exist if $n \geq 2$. Starting from the subset of $\mathcal{B}_{n-2}$ with RSCs 2, 3, ..., $a_{n-2} - 1$, by (3.1) the matrices in $\mathcal{B}_n$ are obtained with RSCs 4, 5, ..., $2(a_{n-2} - 1) + 2$; Therefore, $a_n \geq 2a_{n-2} + 1$. Iterating this inequality, we obtain

$$a_n + 1 \geq 2(a_{n-2} + 1) \geq 2^2(a_{n-4} + 1) \geq \cdots \geq 2^k(a_{n-2k} + 1)$$

for all k, $2k < n$. For $n = 2m \geq 8$ we have $a_n \geq 2^{m-4}(a_8 + 1) - 1 = \frac{55}{8}\sqrt{2}^{\,n} - 1$, and for $n = 2m + 1 \geq 9$ we obtain $a_n \geq 2^{m-4}(a_9 + 1) - 1 = 6\sqrt{2}\sqrt{2}^{\,n} - 1$. In both cases we have $a_n \geq 6\sqrt{2}\sqrt{2}^{\,n} - 1$ for $n \geq 9$. $\square$

Now we show how more equalities of the type (3.1) can be obtained, leading to better lower bounds of the type $a_n > \gamma q^n$, $q > \sqrt{2}$.

Lemma 3.2. *If A has no zero rows and no zero columns then*

$$\mathrm{r}\left[\begin{array}{c|cc} A & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right] = (\mathrm{r}(A) - 2)\,\mathrm{r}(C) + \mathrm{r}\left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right],$$

$$\mathrm{r}\left[\begin{array}{c|cc} A & 0 & 1 \\ \hline 1 & E & F \end{array}\right] = \mathrm{r}(A) - 2 + \mathrm{r}\left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 1 & E & F \end{array}\right],$$

$$\mathrm{r}\left[\begin{array}{c|c} A & 1 \\ \hline 0 & D \\ 1 & F \end{array}\right] = \mathrm{r}(A) - 2 + \mathrm{r}\left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & D \\ 1 & F \end{array}\right], \text{ and}$$

$$\mathrm{r}\left[\begin{array}{c|c} A & 1 \\ \hline 1 & F \end{array}\right] = \mathrm{r}(A) - 2 + \mathrm{r}\left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & F \end{array}\right].$$

The upper left 1 on the right hand sides is the 1×1 matrix.

Proof. Only the first equality has to be proved. The other three are obtained from the first by taking $C = D = 0$, $C = E = 0$, and $C = D = E = 0$, respectively. In each case, the zero matrices may be left out since they do not affect the cardinality of the row space.

Let $A \in \mathcal{B}_{kl}$. Obviously, $\mathrm{r}(A) \geq 2$; the inequality $\mathrm{r}(A) > 2$ is equivalent to $A \neq 1_{k\times l}$. The case $\mathrm{r}(A) = 2$ is trivial: it is enough to remove repeated rows and columns in A, replacing A by $1_{1\times 1}$. Suppose therefore $\mathrm{r}(A) > 2$. The matrix A has no zero columns, and consequently $\mathrm{R}(A)$ contains 1-row, i.e. $[1_{1\times l}\ 0\ 1] = [1\ 0\ 1] \in \mathrm{R}[A\ 0\ 1]$. Denote $\mathcal{W} = \mathrm{R}[A\ 0\ 1] \setminus \{[0\ 0\ 0], [1\ 0\ 1]\}$. Let

$$B = \left[\begin{array}{c|cc} A & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right] \quad \text{and} \quad B' = \left[\begin{array}{c|cc} 1_{k\times l} & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right].$$

Furthermore, let $\mathcal{F}_0 = \mathrm{R}[0\ C\ D]$, $\mathcal{F} = \mathrm{R}\begin{bmatrix} 0 & C & D \\ 1 & E & F \end{bmatrix}$, and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. The first and the third block in elements of $\mathcal{F}_1 + \mathcal{W}$ are 1, and the second block is from $R\begin{bmatrix} C \\ E \end{bmatrix}$, implying

$$\mathcal{F}_1 + \mathcal{W} \subset R\begin{bmatrix} 0 & C & D \\ 1 & E & F \end{bmatrix} + [1\ 0\ 1] = \mathcal{F} + [1\ 0\ 1] \subset \mathrm{R}(B').$$

Therefore,

$$\begin{aligned} R(B) &= ((\mathcal{F} + [0\ 0\ 0]) \cup (\mathcal{F} + [1\ 0\ 1])) \cup (\mathcal{F} + \mathcal{W}) = \mathrm{R}(B') \cup (\mathcal{F} + \mathcal{W}) \\ &= \mathrm{R}(B') \cup (\mathcal{F}_0 + \mathcal{W}) \cup (\mathcal{F}_1 + \mathcal{W}) = \mathrm{R}(B') \cup (\mathcal{F}_0 + \mathcal{W}). \end{aligned} \tag{3.2}$$

The first block in elements of $\mathrm{R}(B')$ is 0 or 1; the first block in elements of $\mathcal{F}_0 + \mathcal{W}$ is never 0 or 1. Consequently, $(\mathcal{F}_0 + \mathcal{W}) \cap \mathrm{R}(B') = \emptyset$ and

$$\mathrm{r}(B) = \mathrm{r}(B') + |\mathcal{F}_0 + \mathcal{W}|.$$

The third block in elements of $\mathcal{W}$ is 1; hence, the same is true for all elements of $\mathcal{F}_0 + \mathcal{W}$. Therefore

$$|\mathcal{F}_0 + \mathcal{W}| = |\,\mathrm{R}[0\ C\ D] + \mathcal{W}| = |\,\mathrm{R}[0\ C\ 1] + \mathcal{W}| = \mathrm{r}(C)|\mathcal{W}| = (\mathrm{r}(A) - 2)\,\mathrm{r}(C),$$

implying

$$\mathrm{r}(B) = \mathrm{r}(B') + (\mathrm{r}(A) - 2)\,\mathrm{r}(C). \tag{3.3}$$

The first k rows of B' are identical, and so are the first l columns. Keeping only one of these identical rows and columns, we complete the proof of the theorem. $\square$

According to Lemma 3.2, $\mathrm{r}(B)$ linearly depends on $\mathrm{r}(A)$, with the multiplier $\mathrm{r}(C)$ (or 1) and with the free coefficient

$$\mathrm{r}\begin{bmatrix} 1 & 0 & 1 \\ 0 & C & D \\ 1 & E & F \end{bmatrix} - 2\,\mathrm{r}(C) \geq \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - 2\,\mathrm{r}(C) = 0.$$

The restraint that A has no zero rows and no zero columns is not critical: if $A = \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix}$, then

$$B = \mathrm{r}\begin{bmatrix} A & 0 & 1 \\ 0 & C & D \\ 1 & E & F \end{bmatrix} = \mathrm{r}\left[\begin{array}{c|cc|c} A' & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & C & D \\ \hline 1 & 1 & E & F \end{array}\right] = \mathrm{r}\begin{bmatrix} A' & 0 & 1 \\ 0 & C' & D' \\ 1 & E' & F \end{bmatrix},$$

where $C' = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$, $D' = \begin{bmatrix} 1 \\ D \end{bmatrix}$, and $E' = [1\ E]$. This makes it possible to apply Lemma 3.2 to the matrix B.

The statement of Lema 3.2 can be reformulated. Let $A \in \mathcal{B}_{kl}$ be a matrix with no zero rows and no zero columns; let $x \in \mathcal{B}_{r1}$, $y \in \mathcal{B}_{1s}$, and

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad C = [x\ x\ \cdots\ x], \quad B = \begin{bmatrix} y \\ y \\ \vdots \\ y \end{bmatrix}.$$

If $x \neq 1_{r\times 1}$ and $y \neq 1_{1\times s}$, and F is the matrix obtained from D by eliminating the rows corresponding to ones in x and the columns corresponding to ones in y, then

$$\mathrm{r}(G) = (\mathrm{r}(A) - 2)\,\mathrm{r}(F) + \mathrm{r}\begin{bmatrix} 1 & y \\ x & D \end{bmatrix}.$$

Otherwise, if If $x = 1_{r\times 1}$ or $y = 1_{1\times s}$, then

$$\mathrm{r}(G) = \mathrm{r}(A) - 2 + \mathrm{r}\begin{bmatrix} 1 & y \\ x & D \end{bmatrix}.$$

Example 3.3. If A has no zero rows and no zero columns and $\mathrm{r}(A) = a \neq 0$, then

$$\begin{aligned} \mathrm{r}\left[\begin{array}{c|cc|c} A & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 \end{array}\right] &= (\mathrm{r}(A) - 2) * \mathrm{r}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \mathrm{r}\left[\begin{array}{c|cc|c} 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 \end{array}\right] \\ &= (\mathrm{r}(A) - 2) * 3 + 7 = 3a + 1. \end{aligned}$$

Similarly,

$$\mathrm{r}\left[\begin{array}{c|cc|c} A & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 \end{array}\right] = 3a, \quad \mathrm{r}\left[\begin{array}{c|cc|c} A & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \end{array}\right] = 3a+2.$$

This is not quite a random set of examples: it is similar to (3.1); it can be further generalized. Let $m \geq 2$, $a \geq 1$, $b \geq a-1$ and suppose that for each c, $0 \leq c \leq b$ there exists a matrix

$$U_{a,c} = \begin{bmatrix} C_{a,c} & D_{a,c} \\ E_{a,c} & F_{a,c} \end{bmatrix} \in \mathcal{B}_m$$

such that

$$\mathrm{r}\left[\begin{array}{c|cc} A & 0 & 1 \\ \hline 0 & C_{a,c} & D_{a,c} \\ 1 & E_{a,c} & F_{a,c} \end{array}\right] = a\,\mathrm{r}(A) + c$$

for all $A \in \mathcal{B}_n$, A without any zero rows or columns. We will call a collection $\{U_{a,c} \mid 0 \leq c \leq b\}$ an (m, a, b) *system.*

We now prove that the existence of an (m, a, b) system implies the existence of a lower bound of a_n of the form $c\,\sqrt[m]{a}^{\,n}$.

Theorem 3.4. *Let $m \geq 2$. Suppose that for some $a \geq 1$, $b \geq a-1$ there exists an arbitrary (m, a, b) system. Suppose that for some $k \geq 2$ we have $a_k \geq 2a$. Let $\alpha = (b+1-a)/(a-1)$, $q = \sqrt[m]{a}$, and*

$$\gamma = \min\left\{(a_{k+i} + \alpha)q^{-(k+i)} \mid 0 \leq i < m\right\}.$$

Then for all $n \geq k$ we have $a_n \geq \gamma q^n - \alpha$.

Proof. The inequalities $a_{k+i} \geq \gamma q^{k+i} - \alpha$, $0 \leq i < m$, follow from definition of γ, i.e. the statement of the theorem is true for m consecutive integers $i = k, k+1, \ldots, k+m-1$. Suppose the claim is true for all n, $k \leq n < N$ ($N > k+m-1$), and let $n = N$.

From $a_k \geq 2a$ and $n \geq k$ it follows $[1, 2a-1] \subset \mathcal{R}_n$.

Let $A_i \in \mathcal{B}_{n-m}$ be square matrices satisfying $\mathrm{r}(A_i) = i$, $2 \leq i \leq a_{n-m} - 1$. Without the loss of generality, we can suppose that A_i are without zero rows and zero columns: an arbitrary matrix $A \neq 0$ with zero rows and/or zero columns can be replaced with A', $\mathrm{r}(A) = \mathrm{r}(A')$, obtained from A by replacing zero rows (columns) by copies of some non-zero rows (columns). Let

$$U_{a,c} = \begin{bmatrix} C_{a,c} & D_{a,c} \\ E_{a,c} & F_{a,c} \end{bmatrix}, \quad 0 \leq c \leq b,$$

be the matrices contained in an (m, a, b) system. Then

$$\bigcup_{i=2}^{a_{n-m}-1} \bigcup_{c=0}^{b} \left\{ \mathrm{r}\left[\begin{array}{c|cc} A_i & 0 & 1 \\ \hline 0 & C_{a,c} & D_{a,c} \\ 1 & E_{a,c} & F_{a,c} \end{array}\right] \right\} = \bigcup_{i=2}^{a_{n-m}-1} \bigcup_{c=0}^{b} \{ia + c\} = [2a, aa_{n-m} + b - a],$$

implying $[2a, aa_{n-m} + b - a] \subset \mathcal{R}_n$, and

$$a_n \geq aa_{n-m} + b - a + 1. \tag{3.4}$$

From the inductive hypothesis it follows

$$a_n \geq a\left(\gamma q^{n-m} - \alpha\right) + b - a + 1 = \gamma q^n - \alpha.$$

TABLE 2. The triplets (m, a, b) for which we found (m, a, b) systems.

m	a	b	$q = \sqrt[m]{a}$	$\alpha = \frac{b+1-a}{a-1}$
2	2	2	1.41421	1
3	3	4	1.44225	1
4	6	6	1.56508	1/5
5	10	10	1.58489	1/9
6	18	18	1.61887	1/17
7	30	32	1.62561	3/29
8	56	60	1.65395	1/11
9	102	114	1.67177	13/101
10	193	218	1.69261	13/96
11	336	350	1.69694	3/67

Therefore, the theorem is proved by induction. □

If we cannot determine a_n for some n, then it is useful to know any lower bound $\bar{a}_n \le a_n$.

Note. The condition $a_n \ge 2a$ is not crucial, because by Theorem 3.1 it is satisfied for all $n \ge 2\log_2((2a+1)/(6\sqrt{2}))$. Therefore, the existence of a (m, a, b) system implies the lower bound of the form $a_n \ge \gamma \sqrt[m]{a}^{\,n} - \alpha$. The constant γ is estimated using m consecutive lower bounds $\bar{a}_n \le a_n$; if we replace a_{k+i} by $\bar{a}_{k+i}$ in the definition of γ, we obtain a lower bound worse only by a constant factor. In the next section we demonstrate how to find good lower bounds $\bar{a}_n$ using a generalization of Lemma 3.2.

From Lemma 3.2 we see that the coefficient $a = \mathrm{r}(C)$ depends only on the matrix C. In order to search for an (m, a, b) system, for given a and m, we start from a set of matrices C, satisfying $\mathrm{r}(C) = a$. To reduce the search space, it is chosen to search for C among matrices of order $m-1$, i.e. E is a row, and D is a column vector. By varying matrices D, E and F, some set of coefficients c is obtained, possibly constituting a complete (m, a, b) system if $b \ge a - 1$. In Table 2 the triplets (m, a, b) are shown, for which we found (m, a, b) system, $2 \le m \le 11$. The triplets are accompanied by $q = \sqrt[m]{a}$ and $\alpha = (b+1-a)/(a-1)$. The best lower bound is obtained for $(m, a, b) = (11, 336, 350)$.

The part of (m, a, b) systems mentioned in Table 2 for $m \le 6$, is shown in Table 3. The rows of $U_{a,c}$ are represented by hexadecimal numbers, as in Table 1); dimensions of $C_{a,c}$ are 1×1. The matrices from Example 3.3 are the part of the $(3, 3, 4)$ system from Table 3. All the systems found can be seen at http://www.matf.bg.ac.yu/ ezivkovm/RSC.htm.

Combining inequalities (3.4) for all triplets (m, a, b) from Table 2, we obtain the inequality

$$\begin{aligned} a_n \ \ge\ & \max\{2a_{n-2}+1, 3a_{n-3}+2, 6a_{n-4}+1, 10a_{n-5}+1, 18a_{n-6}+1, 30a_{n-7}+3, \\ & 56a_{n-8}+5, 102a_{n-9}+13, 193a_{n-10}+26, 336a_{n-11}+15\} \end{aligned} \tag{3.5}$$

which is satisfied if $n \ge 2\log_2((2*336+1)/(6\sqrt{2}))$ (implying $a_n \ge 2*336 = 672$), i.e. if $n \ge 13$.

TABLE 3. Example (m, a, b) systems, $2 \le m \le 6$.

c	$U_{2,c}$ in $(2,2,2)$ system				
0	2	3			
1	3	2			
2	2	0			
c	$U_{3,c}$ in $(3,3,4)$ system				
0	2	6	7		
1	2	6	5		
2	3	6	6		
3	3	6	2		
4	3	6	4		
c	$U_{4,c}$ in $(4,6,6)$ system				
0	2	4	A	F	
1	2	4	A	D	
2	2	4	A	9	
3	3	4	A	E	
4	3	4	A	A	
5	3	4	A	C	
6	3	4	A	2	
c	$U_{5,c}$ in $(5,10,10)$ system				
0	2	4	8	16	1F
1	2	4	8	16	1D
2	2	4	8	16	15
3	2	4	8	16	19
4	3	4	8	16	16
5	3	4	8	16	1C
6	2	4	8	16	11
7	3	5	8	16	18
8	3	4	8	16	18
9	3	5	9	16	10
10	2	4	9	16	10

c	$U_{6,c}$ in $(6,18,18)$ system					
0	2	4	8	10	2E	3F
1	2	4	8	10	2E	3D
2	2	4	8	10	2E	2D
3	2	4	8	10	2E	39
4	3	4	8	10	2E	2E
5	3	4	8	10	2E	3C
6	2	4	8	10	2E	29
7	2	4	8	10	2E	31
8	3	4	8	10	2E	38
9	3	4	8	10	2E	32
10	2	4	8	11	2E	28
11	3	5	9	10	2E	30
12	3	4	8	10	2E	2
13	3	4	8	11	2E	30
14	2	4	8	10	2E	21
15	2	4	8	11	2E	30
16	3	4	8	10	2E	22
17	3	5	9	11	2E	20
18	3	5	8	11	2E	20

4. MORE GENERAL CONSTRUCTION AND IMPROVED LOWER BOUND FOR a_n

We now give a generalization of Lema 3.2. Using this statement, we obtained fairly large subsets of $\mathcal{R}_n$ and sharp lower bounds $\bar{a}_n \le a_n$, $n \le 27$.

Theorem 4.1. *Let*

$$B = \left[\begin{array}{cccc|c} A_1 & 0 & \cdots & 0 & B_1 \\ 0 & A_2 & \cdots & 0 & B_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & A_k & B_k \\ \hline C_1 & C_2 & \cdots & C_k & D \end{array}\right].$$

If A_i has no zero rows and no zero columns, and if the matrix C_i has constant rows (i.e. columns with all elements identical) and if the matrix B_i has constant columns, $1 \le i \le k$, then $\mathrm{r}(B)$ is a multilinear function (i.e. polynomial with exponents not *exceeding* 1*) in terms of* r(A_1), r(A_2), ..., r(A_k)*:*

$$\mathrm{r}(B) = \sum_i \alpha_i \prod_{j=1}^{k} \mathrm{r}(A_j)^{x_{ij}}, \quad x_{ij} \le 1.$$

Proof. We proceed by induction on k. The case $k = 1$ is a consequence of Lemma 3.2. Suppose now $k > 1$. Let B'_i denote the matrix obtained from B_i by removing columns corresponding to 1-columns of B_1, $2 \le i \le k$; if $B_1 = 1$, then B'_i and D' are "empty" matrices (matrices with no columns). Analogously, let C'_i denote the matrix obtained from C_i by removing rows corresponding to 1-rows of C_1, $2 \le i \le k$; if $C_1 = 1$, then C'_i and D' are "empty" matrices (matrices with no rows). Applying reformulated Lemma 3.2 to B and permuting rows and columns, we get

$$\mathrm{r}(B) = (\mathrm{r}(A_1) - 2)\,\mathrm{r}\left[\begin{array}{cccc|c} A_2 & 0 & \cdots & 0 & B'_2 \\ 0 & A_3 & \cdots & 0 & B'_3 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & A_k & B'_k \\ \hline C'_2 & C'_3 & \cdots & C'_k & D' \end{array}\right] + \mathrm{r}\left[\begin{array}{ccc|cc} A_2 & \cdots & 0 & B_2 & 0 \\ & \ddots & & \vdots & \\ 0 & \cdots & A_k & B_k & 0 \\ \hline C_2 & \cdots & C_k & D & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{array}\right].$$

By the inductive hypothesis we conclude that RSCs of these two matrices are (multilinear) polynomials depending on r(A_2),..., r(A_k) — completing the proof. $\square$

From the proof it is seen how the expression for r(B) can be effectively obtained. We now consider some special cases of Theorem 4.1. We suppose that $D \in \mathcal{B}_m$ is quadratic, and consider the cases $m = 0, 1, 2, 3$ and $m > 3$.

The case $m = 0$: Obviously,

$$\mathrm{r}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \mathrm{r}(A_1) * \mathrm{r}(A_2). \tag{4.1}$$

The case $m = 1$**:** Consider the matrix

$$B_1 = \left[\begin{array}{ccc|c} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 1 \\ 0 & 0 & A_3 & 1 \\ \hline 1 & 0 & 1 & D \end{array}\right] \tag{4.2}$$

and all the 7 matrices obtained from B_1 by deleting the rows and columns with the indices from the same subset of $\{1, 2, 3\}$. In Table 4 the polynomials are shown, expressing RSCs of these matrices in terms of r$(A_1) = b$, r$(A_2) = c$, r$(A_3) = d$. For example, if $D = 0$ and we delete the row and column corresponding to A_2, then RSC of the matrix obtained is equal to $1 + bd = 1 + \mathrm{r}(A_1)\,\mathrm{r}(A_3)$. Note that some of these polynomials are equivalent, (they can be made identical by renaming variables). There are here 5 substantially different polynomials: $1 + d$, bc, $1 + bd$, bcd, $1 + bcd$.

TABLE 4. RSCs of submatrices of B_1 (4.2).

Included blocks	$D=0$	$D=1$
A_1, A_2, A_3	$1+bcd$	bcd
A_1, A_2	bc	$1+bc$
A_1, A_3	$1+bd$	bd
A_1	b	$1+b$
A_2, A_3	$1+cd$	cd
A_2	c	$1+c$
A_3	$1+d$	d

TABLE 5. $\mathrm{r}(B_2)$ (4.3) for various $D \in \mathcal{B}_2^\pi$.

D		$\mathrm{r}(B_2) - abcdefghijklmno$
0	0	$1+a+b+d+ade+bdf+h+ahi+bhj$
0	1	$a+d+ade+bdf+ahi$
0	3	$a+ade+ahi$
1	2	$ade+bhj$
1	3	ade
3	3	0

The case $m=2$: Now we consider all $2^{15}-1$ submatrices of

$$B_2 = \left[\begin{array}{ccccccccccccccc|c}
A_1 & & & & & & & & & & & & & & & 0\ 0 \\
& A_2 & & & & & & & & & & & & & & 0\ 0 \\
& & A_3 & & & & & & & & & & & & & 0\ 0 \\
& & & A_4 & & & & & & & & & & & & 0\ 1 \\
& & & & A_5 & & & & & & & & & & & 0\ 1 \\
& & & & & A_6 & & & & & & & & & & 0\ 1 \\
& & & & & & A_7 & & & & & & & & & 0\ 1 \\
& & & & & & & A_8 & & & & & & & & 1\ 0 \\
& & & & & & & & A_9 & & & & & & & 1\ 0 \\
& & & & & & & & & A_{10} & & & & & & 1\ 0 \\
& & & & & & & & & & A_{11} & & & & & 1\ 0 \\
& & & & & & & & & & & A_{12} & & & & 1\ 0 \\
& & & & & & & & & & & & A_{13} & & & 1\ 1 \\
& & & & & & & & & & & & & A_{14} & & 1\ 1 \\
& & & & & & & & & & & & & & A_{15} & 1\ 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \multirow{2}{*}{D} \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 &
\end{array}\right] \tag{4.3}$$

where $D \in \mathcal{B}_2^\pi$. In Table 5 the 6 polynomials, corresponding to various $D \in \mathcal{B}_2^\pi$ are shown with $\mathrm{r}(A_i)$, $1 \le i \le 15$, substituted by $a, b, \ldots, n$ respectively.

As in the previous case, a broader polynomial family is obtained by deleting rows and columns with the indices from the same set $\{1, 2, \ldots, 15\}$. The set of $32767 = 2^{15}-1$ polynomials is reduced by removing equivalent polynomials to a smaller set of 8534 polynomials.

The case $m=3$: Considering this case analogously to the previous case is impossible, because there are $2^{63}-1$ submatrices. Therefore, we decided to start from the following matrix containing 9 diagonal blocks, hoping that representative enough polynomials set will be obtained.

$$B_3 = \left[\begin{array}{ccccccccc|c} A_1 & & & & & & & & & 0\,0\,1 \\ & A_2 & & & & & & & & 0\,0\,1 \\ & & A_3 & & & & & & & 0\,0\,1 \\ & & & A_4 & & & & & & 0\,1\,0 \\ & & & & A_5 & & & & & 0\,1\,0 \\ & & & & & A_6 & & & & 0\,1\,0 \\ & & & & & & A_7 & & & 1\,0\,0 \\ & & & & & & & A_8 & & 1\,0\,0 \\ & & & & & & & & A_9 & 1\,0\,0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & D \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \end{array}\right]. \tag{4.4}$$

Again we computed RSC for each of $2^9 - 1$ submatrices of B_3 and for each of 37 "kernels" $D \in \mathcal{B}_3^\pi$. After removing equivalent polynomials, we are left with 10357 new polynomials. In Table 6 only the polynomials corresponding to the complete matrix B_3 are shown.

The case $m > 3$: Here we considered only matrices with only one diagonal block. As a special case, we obtain various (m, a, b) systems.

Data base containing all polynomials can be found at

http://www.matf.bg.ac.yu/ ezivkovm/RSC.htm. These polynomials are used in Algorithm 4.2 to obtain matrices with various RSCs for $n \leq 27$.

Algorithm 4.2. Generate a subset of $\mathcal{R}'_n \subset \mathcal{R}_n$ starting from a collection P of polynomials and from the subsets $\mathcal{R}'_i \subset \mathcal{R}_i$, $i < n$.

Input :
n — integer,
P — collection of polynomials,
subsets $\mathcal{R}'_i \subset \mathcal{R}_i$, $1 \leq i < n$.
Output : subset $\mathcal{R}'_n \subset \mathcal{R}_n$.

```
R ← ∅;
for m = 0, 11
  for k = 1, 15
    for all p ∈ P
      for all partitions n = m + x_1 + x_2 + ··· + x_k
        for all v_1 ∈ R'_{x_1}, v_2 ∈ R'_{x_2}, ..., v_k ∈ R'_{x_k}
          R ← R ∪ {P(v_1, v_2, ..., v_k)};
          ; retain m, k, p, x_1, x_2, ..., x_k, v_1, v_2, ..., v_k
          ; in order to reconstruct later of the matrix with this RSC
R'_n ← R;
```

Because for large n this starts to be time consuming, the following heuristic is used:

- For $n \leq 22$ the complete collection P is used.
- A subcollection P' is formed, containing all polynomials of degree less than 5, and polynomials of degree at least 5 which resulted in finding at least one new RSC for $n \leq 22$,
- For $23 \leq n \leq 27$ the collection P' is used instead of P.

TABLE 6. $\mathrm{r}(B_3)$ (4.4) for various "kernels" D.

D			$\mathrm{r}(B_3) - abcdefghi$
0	0	0	$13 + a + b + ab + c + ac + bc + d + ad + e + be + de + abde + f + cf + df + acdf + ef + bcef + g + ag + dg + h + bh + eh + gh + abgh + degh + i + ci + fi + gi + acgi + dfgi + hi + bchi + efhi$
0	0	1	$6 + a + b + ab + c + ac + bc + d + ad + be + de + abde + cf + df + acdf + bcef + g + ag + dg + bh + gh + abgh + degh + ci + gi + acgi + dfgi + bchi$
0	0	7	$3 + a + ab + ac + d + ad + de + abde + df + acdf + g + ag + dg + gh + abgh + degh + gi + acgi + dfgi$
1	1	1	$3 + a + b + ab + c + ac + bc + ad + be + abde + cf + acdf + bcef + ag + bh + abgh + ci + acgi + bchi$
7	7	7	0
0	0	3	$4 + a + ab + ac + d + ad + be + de + abde + cf + df + acdf + bcef + g + ag + dg + gh + abgh + degh + gi + acgi + dfgi$
3	3	3	$1 + ad + be + abde + cf + acdf + bcef$
0	1	1	$4 + a + b + ab + c + ac + bc + ad + be + de + abde + cf + acdf + bcef + ag + bh + gh + abgh + degh + ci + acgi + bchi$
0	7	7	$1 + ab + de + abde + gh + abgh + degh$
0	3	3	$2 + ab + ad + be + de + abde + cf + acdf + bcef + gh + abgh + degh$
0	1	2	$2 + a + ab + ac + ad + e + be + de + abde + cf + acdf + ef + bcef + ag + eh + gh + abgh + degh + acgi + efhi$
0	1	7	$1 + a + ab + ac + ad + de + abde + acdf + ag + gh + abgh + degh + acgi$
1	1	3	$1 + a + ab + ac + ad + be + abde + cf + acdf + bcef + ag + abgh + acgi$
1	1	6	$a + ab + ac + ad + abde + acdf + ag + abgh + acgi + efhi$
3	7	7	$abde$
0	1	3	$2 + a + ab + ac + ad + be + de + abde + cf + acdf + bcef + ag + gh + abgh + degh + acgi$
0	1	6	$1 + a + ab + ac + ad + de + abde + acdf + ag + eh + gh + abgh + degh + acgi + efhi$
0	3	7	$1 + ab + ad + de + abde + acdf + gh + abgh + degh$
1	1	2	$1 + a + ab + ac + ad + be + abde + cf + acdf + ef + bcef + ag + abgh + acgi + efhi$
1	3	3	$1 + ab + ad + be + abde + cf + acdf + bcef + abgh$
1	1	7	$a + ab + ac + ad + abde + acdf + ag + abgh + acgi$
3	3	7	$ad + abde + acdf$
1	6	6	$ab + abde + abgh + fi + dfgi + efhi$
1	7	7	$ab + abde + abgh$
0	3	5	$1 + ab + ad + de + abde + acdf + bh + gh + abgh + degh + bchi$
3	3	5	$ad + abde + acdf + bchi$
1	2	3	$1 + ab + ad + be + abde + cf + df + acdf + bcef + abgh + dfgi$
1	6	7	$ab + abde + abgh + dfgi$
1	2	4	$ab + ad + abde + df + acdf + bh + abgh + fi + dfgi + hi + bchi + efhi$
1	2	7	$ab + ad + abde + df + acdf + abgh + dfgi$
1	3	5	$ab + ad + abde + acdf + bh + abgh + bchi$
1	3	6	$ab + ad + abde + acdf + abgh + efhi$
3	5	7	$abde + acgi$
1	2	5	$ab + ad + abde + df + acdf + bh + abgh + dfgi + bchi$
1	3	7	$ab + ad + abde + acdf + abgh$
1	1	2	$1 + a + ab + ac + ad + be + abde + cf + acdf + ef + bcef + ag + abgh + acgi + efhi$
3	5	6	$abde + acgi + efhi$

TABLE 7. The lower bounds $\bar{a}_n \le a_n$ and $|\mathcal{R}_n|$, $n \le 27$; if $n \le 9$ then $\mathcal{R}'_n = \mathcal{R}_n$ and $\bar{a}_n = a_n$.

n	$\bar{a}_n$	$\lvert\mathcal{R}'_n\rvert$	n	$\bar{a}_n$	$\lvert\mathcal{R}'_n\rvert$
1	3	2	15	7537	10024
2	5	4	16	14009	18890
3	7	7	17	24479	35505
4	11	12	18	46583	66643
5	19	21	19	81655	124834
6	35	38	20	146939	232602
7	61	69	21	257759	432531
8	109	126	22	488689	806104
9	191	232	23	962011	1508565
10	363	429	24	1759611	2835495
11	685	799	25	3136799	5348392
12	1235	1494	26	6019681	10115206
13	2271	2808	27	11752769	19163066
14	3959	5309			

In Table 7 the lower bounds $\bar{a}_n \le a_n$, and the sizes $|\mathcal{R}'_n| \le |\mathcal{R}_n|$ are shown, $n \le 27$. Data retained in Algorithm 4.2, sufficient to reconstruct matrices with these RSCs, can be found at http://www.matf.bg.ac.yu/ ezivkovm/RSC.htm.

In Table 8 the lower bounds $\bar{a}_n$, $28 \le n \le 54$ obtained by (3.5) are shown. In order to get a rough picture of the growth rate of $\bar{a}_n$, the values $\log_2 \bar{a}_n - 0.865n$ are also shown in Table 8. The constant $c = 0.865$ is chosen so that $\log_2 \bar{a}_n - cn$ is close to zero as long, as possible. It turns out that after $n = 27$ this difference sharply falls down.

An interesting open question remains about the exact asymptotic of a_n. According to Table 8, it seems that it is possible to find new (m, a, b) systems, with larger $q = \sqrt[m]{a}$.

Now we give a good lower bound for a_n.

Theorem 4.3. *If $n \ge 31$ then $a_n \ge 5\,\sqrt[11]{336}^{\,n}$.*

Proof. This is a consequence of Theorem 3.4, based on a $(11, 336, 350)$ system from Table 2 (http://www.matf.bg.ac.yu/ ezivkovm/RSC.htm). Let $q = \sqrt[11]{336}$. For $k = 31$ we have by the values of $\bar{a}_n$ listed in Table 8

$$\gamma \ge \min\left\{(\bar{a}_{31+i} + \alpha)q^{-31-i} \mid 0 \le i \le 10\right\} := \bar{\gamma} = (\bar{a}_{39} + \alpha)q^{-39} > 5,$$

and therefore, because of $\bar{\gamma} \simeq 5.008486$ and $(\bar{\gamma} - 5)q^{31} \simeq 111783.8$, which is greater than α, we get

$$a_n \ge \bar{\gamma}q^n - \alpha = 5q^n + (\bar{\gamma} - 5)q^n - \alpha \ge 5q^n + (\bar{\gamma} - 5)q^{31} - \alpha \ge 5q^n, \quad n \ge 31.$$

□

5. THE SET $\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}]$

The construction based on Theorem 4.1 made it possible to move towards extending the Konieczny result [4] from $(2^{n-1}, 2^n]$ to the interval $(2^{n-2} + 2^{n-3}, 2^{n-1}]$.

TABLE 8. The lower bounds $\bar{a}_n \le a_n$, $n \le 54$.

n	$\bar{a}_n$	$\log_2 \bar{a}_n$ $-0.865n$	n	$\bar{a}_n$	$\log_2 \bar{a}_n$ $-0.865n$
1	3	0.7200	28	12039363	-0.6987
2	5	0.5919	29	23505539	-0.5985
3	7	0.2124	30	36118087	-0.8438
4	11	-0.0006	31	70516615	-0.7435
5	19	-0.0771	32	117527691	-0.8716
6	35	-0.0607	33	211549843	-0.8886
7	61	-0.1243	34	352583073	-1.0166
8	109	-0.1518	35	658155069	-0.9811
9	191	-0.2076	36	1198782451	-0.9811
10	363	-0.1462	37	2268284443	-0.9260
11	685	-0.0950	38	3948930399	-0.9912
12	1235	-0.1097	39	4536569053	-1.6560
13	2271	-0.0959	40	7897861119	-1.7212
14	3959	-0.1591	41	13609706721	-1.8011
15	7537	-0.0952	42	23693582655	-1.8662
16	14009	-0.0659	43	40829119975	-1.9461
17	24479	-0.1257	44	71080747263	-2.0113
18	46583	-0.0625	45	127023928813	-2.0387
19	81655	-0.1177	46	231365013199	-2.0386
20	146939	-0.1351	47	437778897525	-1.9836
21	257759	-0.1893	48	762143572863	-2.0487
22	488689	-0.1314	49	1326840614079	-2.1139
23	962011	-0.0193	50	1524287201823	-2.7787
24	1759611	-0.0132	51	2653681335999	-2.8439
25	3136799	-0.0441	52	4572861458271	-2.9238
26	6019681	0.0313	53	7961043772095	-2.9889
27	11752769	0.1315	54	13718584311615	-3.0688

Theorem 5.1. *Let*

$$
\begin{aligned}
\mathcal{A}_3 &= \{2^i \mid 0 \le i \le n-4\}, \\
\mathcal{A}_4 &= \{2^i + 2^j \mid 0 \le j < i \le n-4\}, \\
\mathcal{A}'_5 &= \{2^i + 2^{k+1} + 2^k \mid 0 \le k \le n-6,\ k+2 \le i \le n-4\}, \\
\mathcal{A}''_5 &= \begin{cases} \{2^i + 2^j + 2^k \mid n \ge 11,\ 1 \le k \le n-10, \\ \qquad k+2 \le j \le \min\{i-1, n+k-5-i\}, & n \ge 11 \\ \emptyset, & n < 11 \end{cases}, \\
\mathcal{A} &= 2^{n-2} + 2^{n-3} + (\mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}'_5 \cup \mathcal{A}''_5).
\end{aligned}
$$

Then $\mathcal{A} \subset \mathcal{R}_n$ *and*

$$
|\mathcal{A}| = n^2 - 7n + 14 + \frac{(n-8)(n-10)(2n-15) + 3(n \bmod 2)}{24}
$$

holds for $n \ge 7$.

Proof. Denote by T_n the lower triangular matrix from $\mathcal{B}_n$. For $0 \le i \le n-4$ we have

$$\mathrm{r}\begin{bmatrix} T_2 & 0 & 0 & 0 \\ 0 & I_{n-3-i} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & I_i \end{bmatrix} = 2^i(3 * 2^{n-3-i} + 1) = 2^{n-2} + 2^{n-3} + 2^i,$$

and so $2^{n-2} + 2^{n-3} + \mathcal{A}_3 \subset \mathcal{R}_n$

Consider the matrices

$$B_4 = \left[\begin{array}{ccc|cc} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 1 & 0 \\ 0 & 0 & A_3 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right],\ B_5' = \left[\begin{array}{ccc|cc} A_1 & 0 & 0 & 0 & 1 \\ 0 & A_2 & 0 & 0 & 1 \\ 0 & 0 & A_3 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right],\ B_5'' = \left[\begin{array}{cccc|cc} A_1 & 0 & 0 & 0 & 1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & A_4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right].$$

Let $\mathrm{r}(A_1) = a$, $\mathrm{r}(A_2) = b$, $\mathrm{r}(A_3) = c$, $\mathrm{r}(A_4) = d$. Applying recursive procedure from the proof of Theorem 4.1 we obtain

$$\begin{aligned} \mathrm{r}(B_4) &= abc + ab + b + 1, \\ \mathrm{r}(B_5') &= abc + ab + a + c + 1, \\ \mathrm{r}(B_5'') &= abcd + abc + a + b + d. \end{aligned}$$

Inequalities $0 \le j < i \le n-4$ are equivalent to $n-i-3 \ge 1$, $i-j \ge 1$, $j \ge 0$. After replacing A_1, A_2, A_3 in B_4 by I_{n-3-i}, I_{i-j}, I_1, respectively, and by adding diagonal block I_j (if $j \ge 1$), we obtain

$$\begin{aligned} \mathrm{r}\left[\begin{array}{ccc|cc|c} I_{n-3-i} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{i-j} & 0 & 1 & 0 & 0 \\ 0 & 0 & I_1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_j \end{array}\right] &= 2^j(2^{n-2-j} + 2^{n-3-j} + 2^{i-j} + 1) \\ &= 2^{n-2} + 2^{n-3} + 2^i + 2^j. \end{aligned}$$

Therefore, $2^{n-2} + 2^{n-3} + \mathcal{A}_4 \subset \mathcal{R}_n$.

Inequalities $0 \le k < k+1 < i \le n-4$ are equivalent to $i-k \ge 2$, $n-i-3 \ge 1$, $k \ge 0$. After replacing A_1, A_2, A_3 in B_5' by I_{i-k}, I_{n-i-3}, I_1 respectively, and by adding diagonal block I_k (if $k \ge 1$), we obtain

$$\begin{aligned} \mathrm{r}\left[\begin{array}{ccc|cc|c} I_{i-k} & 0 & 0 & 0 & 1 & 0 \\ 0 & I_{n-i-3} & 0 & 0 & 1 & 0 \\ 0 & 0 & I_1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_k \end{array}\right] &= 2^k(2^{n-2-k} + 2^{n-3-k} + 2^{i-k} + 2 + 1) \\ &= 2^{n-2} + 2^{n-3} + 2^i + 2^{k+1} + 2^k. \end{aligned}$$

Therefore, $2^{n-2} + 2^{n-3} + \mathcal{A}_5' \subset \mathcal{R}_n$.

Inequalities defining $\mathcal{A}_5''$ are redundant: from $k+2 \le j \le \min\{i-1, n+k-5-i\}$ it follows $k+2 \le i-1$ and $k+2 \le n+k-5-i$; adding these two inequalities, we obtain $n-10 \ge k$; finally, $n \ge k+10 \ge 11$. Each triple (i,j,k) from the definition of $\mathcal{A}_5''$ satisfies $i-k+1 \ge 3$, $j-k+1 \ge 3$, $n+k-i-j-4 \ge 1$, $k-1 \ge 0$. After replacing A_1, A_2, A_3, A_4 in B_5'' by I_{i-k+1}, I_{j-k+1}, $I_{n+k-i-j-4}$, I_1, respectively, and by adding diagonal block I_{k-1} (if $k-1 \ge 1$) we obtain

$$\begin{aligned} & \mathrm{r}\left[\begin{array}{cccc|cc|c} I_{i-k+1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & I_{j-k+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n+k-i-j-4} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_{k-1} \end{array}\right] \\ = {} & 2^{k-1}(2^{n-1-k} + 2^{n-2-k} + 2^{i-k+1} + 2^{j-k+1} + 2) \\ = {} & 2^{n-2} + 2^{n-3} + 2^i + 2^j + 2^k. \end{aligned}$$

Therefore, $2^{n-2} + 2^{n-3} + \mathcal{A}_5'' \subset \mathcal{R}_n$. The matrix above is defined if $n \ge 9$, but we require $n \ge 11$ in order to make the sets $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5'$ and $\mathcal{A}_5''$ disjoint. Putting all this together, we see that $\mathcal{A} \subset \mathcal{R}_n$.

Obviously, $|\mathcal{A}_3| = n-3$, $|\mathcal{A}_4| = (n-3)(n-4)/2$, and $|\mathcal{A}_5'| = (n-4)(n-5)/2$. By a little more complicated enumeration,

$$|\mathcal{A}_5''| = \frac{(n-8)(n-10)(2n-15) + 3(n \bmod 2)}{24},$$

and so $|\mathcal{A}| = |\mathcal{A}_3| + |\mathcal{A}_4| + |\mathcal{A}_5'| + |\mathcal{A}_5''|$ is obtained. □

Comparing this with [4], we see that $\mathcal{R}_n \cap (2^{n-1}, 2^n]$ consists of integers with exactly two binary ones, while $\mathcal{R}_n \cap (2^{n-2}+2^{n-3}, 2^{n-1}]$ consists (at least) of integers with 3 or 4 binary ones, and some integers with 5 binary ones (more precisely, the integers $2^{n-2}+2^{n-3}+2^i+2^j+2^k$, satisfying $0 \le k \le n-6$, $j = k+1$, $k+2 \le i \le n-4$ or $1 \le k \le n-10$, $k+2 \le j \le \min\{i-1, n+k-5-i\}$).

Because of good agreement with [5], we can state a

Hypothesis: $\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}] = \mathcal{A}$.

6. Acknowledgement

I am greatly indebted to the anonymous referee for many useful comments, especially regarding Lemma 3.2.

References

1. K. H. Kim, *Boolean matrix theory and applications*, Marcel Dekker, Inc., 1982.
2. M. Breen, D. Hume, On the cardinalities of row spaces of Boolean matrices, Semigroup Forum 62 (2001), 331-332.
3. S. Hong, Distribution of cardinalities of row spaces of Boolean matrices of order n, Southeast Asian Bull. Math. 24 (2000), 51-64.
4. J. Konieczny, On cardinalities of row spaces of Boolean matrices, Semigroup Forum 44 (1992), 393-402.
5. L. Zhong, On the contribution of cardinalities of row spaces of Boolean matrices, Linear Algebra Appl. 288 (1999), 187-198.
6. W. Li, M.-C. Zhang, On Konieczny's Conjecture of Boolean matrices, Semigroup Forum 50 (1995), 37-58
7. M.-C. Zhang,S.-F. Hong, H.-B. Kan, On the cardinalities of the row spaces of non-full rank Boolean matrices, Semigroup Forum 59 (1999), 152-154.
8. M. Živković, Classification of small $(0,1)$ matrices, Linear Algebra Appl. 414 (2006), 310-346.

9. J. Williamson, Determinants whose elements are 0 and 1, Amer. Math. Monthly 53 (1946), 427-434.

11000 Beograd, Marka Čelebonovića 61/15, Serbia and Montenegro
E-mail address: ezivkovm@matf.bg.ac.yu