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THE NEKHOROSHEV THEOREM AND LONG–TERM STABILITIES IN THE SOLAR SYSTEM

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SUMMARY: The Nekhoroshev theorem has been often indicated in the last decades as the reference theorem for explaining the dynamics of several systems which are stable in the long–term. The Solar System dynamics provides a wide range of possible and useful applications. In fact, despite the complicated models which are used to numerically integrate realistic Solar System dynamics as accurately as possible, when the integrated solutions are chaotic the reliability of the numerical integrations is limited, and a theoretical long–term stability analysis is required. After the first formulation of Nekhoroshev's theorem in 1977, many theoretical improvements have been achieved. On the one hand, alternative proofs of the theorem itself led to consistent improvements of the stability estimates; on the other hand, the extensions which were necessary to apply the theorem to the systems of interest for Solar System Dynamics, in particular concerning the removal of degeneracies and the implementation of computer assisted proofs, have been developed. In this review paper we discuss some of the motivations and the results which have made Nekhoroshev's theorem a reference stability result for many applications in the Solar System dynamics.

Key words. Solar System: general – celestial mechanics – chaos

1. INTRODUCTION

The models which are currently used to represent the Solar System dynamics as accurately as possible are very complicated. Modern ephemerides of major and minor planets are obtained from numerical integrations including many effects, such as gravitational interactions with a large number of asteroids, post Newtonian corrections due to general relativity and, where applicable, also small non-gravitational contributions, such as those acting on comets near their perihelion. Numerical integrations reveal that many of these dynamics are chaotic (for a sample of the many articles about chaotic dynamics in the Solar System see: Sussman and Wisdom 1988, 1992, Laskar 1990, Murray and Holman 1999, Guzzo 2005, 2006, Wayne et al. 2010 for the detection of chaos in the planetary orbits; Wisdom 1983, Wisdom and Peale 1984, Laskar and Robutel 1993 for chaotic evolutions of the obliquity of planets and satellites; Wisdom 1982, Milani and Nobili 1992, Murray et al. 1998, Nesvorny and Morbidelli 1998 for chaos in the asteroid belt), with typical Lyapunov times, which measure the reliability of the ephemerides, which span from few years for certain small bodies, such as the Jupiter family comets, up to many Myrs for the planetary orbits. Is it therefore still meaningful to speak of long–term stabilities in the Solar System? The answer to this question is positive, since the orbits of the planets and of most asteroids are much more stable than their characteristic Lyapunov time, a situation described as 'stable chaos' in Milani and Nobili (1992). This fact is typical for a class of mechanical systems, called quasi–integrable Hamiltonian systems, whose Hamiltonian satisfy certain transversality conditions. By considering only the cases for which the non-gravitational forces can be neglected, and since the most relevant effect of general relativity is a small change to the frequency of precession of the perihelion (sometimes with relevant impact on the long-term dynamics, see Laskar 2008), we remain with a Hamiltonian system which is a perturbation of an integrable system, the Kepler problem.

The long–term stability of quasi-integrable Hamiltonian systems has been extensively studied with the methods of Hamiltonian perturbation theory, and the most important results are the KAM (Kolmogorov 1954, Arnold 1963, Moser 1958) and Nekhoroshev's theorems (Nekhoroshev 1977, 1979). Under suitable hypotheses, the KAM theorem proves perpetual stability of a large measure set of initial conditions; Nekhoroshev's theorem proves the long–term stability for open sets of initial conditions, including the resonant chaotic motions. The possibility of proving the long–term stability of chaotic motions is particularly interesting for the applications to the Solar System dynamics. The stability argument of Nekhoroshev's theorem grants stability times, which we denote by T_{N}, which are much longer than the typical Lyapunov times T_{L} of the resonant motions, and increase exponentially with an inverse power of the perturbing parameter.

The Nekhoroshev stability time T_{N} can be directly identified as a lower estimate of the time interval of effective stability of a system. Estimates of the Nekhoroshev stability time T_{N} for simplified models of the Solar System, obtained through numerical computations of resonant or non-resonant normal forms, have been used to explain the stability of the considered systems for times comparable to the age of the Solar System (Giorgilli et al. 1989, Celletti and Ferrara 1996, Giorgilli and Skokos 1997, Skokos and Dokoumetzidis 2001, Efthymiopoulos and Sándor 2005, Lhotka et al. 2008, Giorgilli et al. 2009, Sansottera et al. 2013). The identification of the Nekhoroshev stability time T_{N} also as *the stability time interval* of a system requires to prove that a diffusion of the action variables generically occurs on longer time intervals. All instabilities of the action variables occurring over time intervals larger than T_{N} are usually referred as Arnold diffusion, from the pioneering paper by Arnold (Arnold 1964). For Hamiltonian systems satisfying the hypotheses of Nekhoroshev's theorem the existence of unstable orbits is a non–trivial, in many aspects still open, problem. Already a few years after the numerical detection of chaotic motions by Hénon and Heiles 1964, many studies of numerical diffusion through resonances have been published (Chirikov 1971, Laskar 1993, Efthymiopoulos et al. 1998, Wood et al. 1990, Lichtenberg and Aswani 1998, Konishi and Kaneko 1990) up to the recent papers by our group. In the series of papers (Lega et al. 2003, Guzzo et al. 2005, Froeschlé et al. 2005, Guzzo et al. 2011) we provided numerical evidence for the generic existence of sets of orbits diffusing through the resonances of quasi-integrable systems with statistical and topological properties which are compatible with the description of the dynamics obtained from the resonant normal forms considered in the proof of the Nekhoroshev's theorem (see Section 4). In fact, for small values of perturbation parameter ε, the resonances of the system are organized as a web, the so– called Arnold web, which occupies a small measure set of the phase–space. Only a small fraction of initial conditions in the Arnold web produces chaotic motions, and therefore has the chance to diffuse in the action-space. The long-term diffusion occurs *along* the resonances, and for times smaller than T_{N} it is masked by the larger oscillations occurring on the fast-drift planes. The possibility of measuring the very slow diffusion along the resonances is due to a technique, developed using the fast Lyapunov indicator, which allows to filter the large short–term oscillations on the fast drift planes. The slow diffusion along the resonance has been quantified by a diffusion coefficient D. We found that D scales faster than power laws through many orders of magnitude of the perturbation parameter, compatibly with an exponential law suggested by Nekhoroshev's theorem.

Using similar model systems, estimates of T_{N} and an analysis of its relation with the diffusion coefficient of the action variables have been investigated in Efthymiopoulos (2008), Efthymiopoulos and Harsoula (2013), through the numerical computation of resonant normal forms. These analyses confirm the existence of correlations between the Nekhoroshev stability time and the time of long–term diffusion of the action variables. The relation between Nekhoroshev's theorem and the phase-space diffusion has been studied also in Cincotta et al. (2014).

Finally, we note that direct applications of Nekhoroshev's theorem to the systems related to Solar System stabilities are complicated by the fact that the small parameters are not extremely small, and the Hamilton functions are highly degenerate. Therefore, either straightforward applications of Nekhoroshev's theorem in its original formulations are not possible, or they provide poor results. It is therefore necessary to remove the degeneracies with *ad hoc* arguments, and to compute the threshold value for the perturbing parameter in an efficient way, possibly using computer assisted methods.

In my opinion, an extensive application of the theorem to the asteroid belt could be paradigmatic. Due to the increasing number of discovered asteroids, the asteroid main belt will represent the ideal system for the application of the theorem in all its aspects. In fact, the large number of asteroids provides a large sample of different initial conditions, with a large spectrum of dynamical behaviours: the Kirkwood gaps, the asteroids families, the three–body resonances, and the secular resonances represent very different dynamics with very different related stability times. A meaningful application of the theorem has to be consistent with, and explain, all these situations. In particular, the computation of the Nekhoroshev stability time in all the parts of the main belt can provide the long–term stability of the asteroids well beyond the validity of numerical integrations, which is limited to few Lyapunov times T_{L}. This scientific project started about twenty years ago (Morbidelli and Guzzo 1997, Guzzo and Morbidelli 1997, Guzzo et al. 2002a, Pavlović and Guzzo 2008), and much can still be done (see Section 5). New perspectives have been opened also by the recent theoretical improvements of the stability estimates of Nekhoroshev's theorem and the improvements in the experimental detection of new asteroids.

In Section 2 we provide a description of the statement of the theorem; Section 3 is devoted to steepness; in Section 4 we provide a sketch of the proof; in Section 5 we review some of the most important applications of the theorem which have been performed up to now.

2. THE NEKHOROSHEV THEOREM

Let us consider a real-analytic Hamiltonian system with Hamiltonian given, in standard action-angle variables, by:

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi), \qquad (I,\varphi) \in B \times \mathbb{T}^n \ , \quad (1)$$

where $B \subseteq \mathbb{R}^n$ is an open set domain of the actions; $\mathbb{T}^n$ is the standard n–dimensional torus and ε is a small parameter. The unperturbed Hamiltonian $h(I)$ is integrable and its Hamilton equations are immediately solved: its motions are characterized by constant actions, and the angles evolve linearly with time with frequency vector $\omega(I) = \nabla h(I)$. When $\varepsilon \neq 0$ the perturbation $f(I,\varphi)$ generically breaks the integrability already at first order in ε: without further assumptions on h and f, we are not able to exclude instabilities $I(t) - I(0) \sim 1$ already in the relatively short time intervals of order $T \sim 1/\varepsilon$. Instead, if the function $h(I)$ satisfies a geometric condition, called by Nekhoroshev "steepness", the stability time improves dramatically to an exponential order in $1/\varepsilon$ (Nekhoroshev 1977, 1979). Precisely, Nekhoroshev proved that, under the assumption of steepness, there exist positive constants a, b and ε_* such that for any $0 \leq \varepsilon < \varepsilon_*$ the solutions (I_t, φ_t) of the Hamilton equations for $H(I,\varphi)$ satisfy:

$$|I_t - I_0| \leq \varepsilon^b$$

for any time t satisfying:

$$|t| \ \leq \ \frac{1}{\varepsilon}\ \exp\Big(\frac{1}{\varepsilon^a}\Big).$$

Furthermore, a and b can be taken as follows:

$$a = \frac{2}{12\zeta + 3n + 14} \quad , \quad b = \frac{3a}{2\delta_{n-1}} , \qquad (2)$$

where:

$$\zeta = \Big[\delta_1\Big(\delta_2\Big(\dots(\delta_{n-3}(n\delta_{n-2} + n - 2) + n - 3) + \dots\Big) + 2\Big) + 1\Big] - 1,$$

and the $\delta_1, \dots, \delta_{n-1}$ are the so-called steepness indices of h (see Section 3 for details).

The Nekhoroshev stability time can be identified by:

$$T_{\mathrm{N}} = \frac{1}{\varepsilon}\ \exp\Big(\frac{1}{\varepsilon^a}\Big),$$

and its validity is limited for $\varepsilon < \varepsilon_*$. The two most important quantities related to long-term stability are the value of the so–called stability exponent a, and the threshold value of the small parameter ε_*. In particular, the stability exponent a depends only on the number n of degrees of freedom and on the steepness indices of h; in particular it depends only on the integrable approximation $h(I)$. The threshold value for the perturbation parameter ε_* depends also on analytic properties of the perturbation f. Both values of a and ε_* have been improved from the 1977 formulation of Nekhoroshev's theorem. The fact that the stability exponent a depends only on the number of degrees of freedom n and on values of steepness indices δ_i could be optimal. As a matter of fact, at p. 9 of Nekhoroshev's paper (Nekhoroshev 1977), it is conjectured that the steepness indices of h characterize the long-term stability of H, and it is suggested to investigate also the problem numerically. More precisely, it is conjectured that, among the systems with the same number of degrees of freedom, the systems for which $h(I)$ have smaller steepness indices are more stable than those with larger indices.

In particular, according to this conjecture, since the steep functions with the smallest steepness indices $\delta_1 = ... = \delta_{n-1} = 1$ are the convex (or quasi-convex) ones, perturbations of the convex (or quasi-convex) $h(I)$ are the most stable Hamiltonian systems; a first numerical confirmation of the conjecture has been obtained in Guzzo et al. (2011).

Several proofs of Nekhoroshev's theorem improved the value of the stability exponent a. Most of these improvements concerned the quasi–convex case (Benettin et al. 1985, Lochak 1992, Lochak and Neishtadt 1992, Pöschel 1993, Bounemoura and Marco 2011). The quasi-convexity of a function $h(I)$ is defined by the fact that the only solution $u = (u_1, \dots, u_n)$ of the equations:

$$\sum_{i=1}^{n} \frac{\partial h}{\partial I_i} u_i = 0 \quad , \quad \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial I_i \partial I_j} u_i u_j = 0$$

is $u = (0, \dots, 0)$. Concerning the generic steep case, the improvement of the stability exponent has been recently obtained in Guzzo et al. (2014a, 2014b). The new values for a and b are:

$$a = \frac{1}{2n\delta_1\delta_2 \dots \delta_{n-2}} \quad , \quad b = \frac{a}{\delta_{n-1}}. \qquad (3)$$

The improvement of Eq. (3) with respect to Eq. (2) is evident in the genuine steep non–convex case: in particular a^{-1} grows quadratically with n in Eq.

(2) and only linearly with n in Eq. (3). The long–term stability of model systems with steepness indices equal to 1 and 2 has been numerically investigated in Guzzo et al. (2011), Todorović et al. (2011). The conclusions of these numerical investigations are in agreement with Nekhoroshev's conjecture.

Another parameter which is crucial for the applications of the theorem is the threshold value ε_* for the perturbation parameter ε. The explicit expressions of ε_* obtained from various statements of the theorem provide values which, when applied to Solar System dynamics, are very small, and can be hardly compared to the existent perturbations. The smallness of the threshold parameters is not specific of Nekhoroshev's theorem, but is common to most theorems of perturbation theory, including the KAM theorem (see, e.g. Celletti and Chierchia 1995, 1997). As a matter of fact, the definition itself of the perturbation parameter is not unique: in fact, given a Hamiltonian system, there is not a unique way of writing it as perturbation of an integrable $h(I)$. It may happen that, using an alternative set J, ψ of action–angle variables, usually defined with a finite number of averaging transformations- i.e. Birkhoff normalization steps- the Hamiltonian (1) is reduced to a normal form:

$$H = k(J;\varepsilon) + \varepsilon^N r(J,\psi;\varepsilon)\,, \tag{4}$$

which is a perturbation of $k(J;\varepsilon)$ with perturbation parameter $\mu = \varepsilon^N$ which is much smaller than ε. The construction of preliminary normalizing transformations is strictly related to the specific expansion of the perturbation on a specific domain. In fact, due to the well-known D'Alembert characteristics, the non zero terms of the Fourier expansions of the perturbations of typical problems of Celestial Mechanics are strongly reduced, and some resonances may appear only to higher orders in the perturbation parameter. For example, as discussed in Morbidelli and Guzzo (1997), it is expected that in the different parts of the Main Belt different preliminary averaging transformations may be defined, thus justifying very different Nekhoroshev stability times.

3. THE STEEPNESS PROPERTY

The definition of the steepness of a C^1 function $h(I)$ given by Nekhoroshev can be formulated as follows:

Definition. *$h(I)$ is steep in the set $B \subseteq \mathbb{R}^n$ with steepness indices $\delta_1, \ldots, \delta_{n-1} \geq 1$ and steepness coefficients $C_1, \ldots, C_{n-1}$ and ξ_*, if:*

$$\inf_{I \in B} \|\omega(I)\| > 0$$

where $\omega(I) := \nabla h(I)$ and, for any $I \in B$, and for any j–dimensional linear subspace $\lambda \subset \mathbb{R}^n$ orthogonal to $\omega(I)$ with $1 \leq j \leq n-1$, one has

$$\max_{0 \leq \eta \leq \xi} \ \min_{u \in \lambda : \|u\| = \eta} \|\pi_\lambda \omega(I+u)\| \geq C_j \xi^{\delta_j} \quad \forall\, \xi \in (0, \xi_*], \tag{5}$$

where π_λ denotes the orthogonal projection of a vector over the space λ.

Since to understand Eq. (5) we have to think of its use in estimating the small divisors related to the different resonances, we need to know something about the proof of the theorem. Therefore, we will give more details about the geometric meaning of steepness in Section 4.

The definition given above is of implicit type, while the ideal situation would be to verify the steepness of a given function through the solvability of explicit algebraic equalities and inequalities. For example, it is easy to check that if a function $h(I)$ is quasi–convex in I, i.e. if $\nabla h(I) \neq 0$ and the only solution $u = (u_1, \ldots, u_n)$ of the system:

$$\sum_{i=1}^n \frac{\partial h}{\partial I_i} u_i = 0 \quad , \quad \sum_{i,j=1}^n \frac{\partial^2 h}{\partial I_i \partial I_j} u_i u_j = 0 \tag{6}$$

is $u = (0, \ldots, 0)$, then h is steep with steepness indices $\delta_1 = \ldots = \delta_{n-1} = 1$. The verification of steepness of functions which are not quasi–convex requires to consider also the higher order derivatives, at least of order 3. Nekhoroshev indicated a sufficient condition for steepness based on the solvability of a system of equalities involving the third order derivatives of h. Precisely, if $\nabla h(I) \neq 0$ and the only solution $u = (u_1, \ldots, u_n)$ of the system:

$$\begin{array}{l} \sum_{i=1}^n \frac{\partial h}{\partial I_i} u_i = 0 \\ \sum_{i,j=1}^n \frac{\partial^2 h}{\partial I_i \partial I_j} u_i u_j = 0 \\ \sum_{i,j,k=1}^n \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k} u_i u_j u_k = 0 \end{array} \tag{7}$$

is $u = (0, \ldots, 0)$, then h is steep at I (the steepness indices may be equal to 1 or 2). A function satisfying this condition is also called three-jet non degenerate at I, being the r-jet of a function identified as the collection of all its derivatives up to order r. Quasi-convexity and three–jet non degeneracy are the only known explicit conditions for steepness which are valid for any number n of the degrees of freedom. Nevertheless, they are generic conditions only for $n \leq 3$ (see, for example, the discussion in Schirinzi and Guzzo 2015). For $n = 4$ there exists an explicit condition for steepness proved in Schirinzi and Guzzo (2014), from former general results (Nekhoroshev 1973, 1979), which is generic and formulated using also the derivatives of order four of $h(I)$. For $n \geq 5$, the formulation of generic explicit conditions for testing the steepness of a function is still an open problem.

Which is the steepness property to consider for the Hamiltonians which are used in studies of the Solar System dynamics? The simplest way of writing the Hamiltonian representing some Solar System dynamics in quasi–integrable form is to consider it as a perturbation of one, or more, decoupled Kepler problems and to represent it with the Delaunay action–angle variables L, G, H, l, g, h. With such a choice, the integrable Hamiltonian h depends only on the action variable(s) L, and therefore it is not steep. When the integrable Hamiltonian does not depend on one or more action variables the lack of steepness is called 'proper degeneracy'. Beyond the Kepler problem, another remarkable example of properly degenerate system, useful for study the rotation of celestial bodies, is represented by the Euler–Poinsot rigid body. Perturbations of properly degenerate systems may exhibit large scale chaotic motions on time scales of order $1/\varepsilon$, as it is the case for simplified models of diffusion in the asteroid 3/1 mean motion resonance (see Wisdom 1982, Neishtadt 1987) and of spin-orbit rotations (see Benettin et al. 2008), despite some of the action variables my remain stable up to exponentially long times (see Benettin and Fassó 1996, Benettin et al. 1997, 2004, 2006). By excluding few low order resonances, and by performing a preliminary averaging transformation (as indicated in Section 2), the degeneracy may possibly be eliminated (see Guzzo and Morbidelli 1997, Guzzo 1999). For example, outside low order mean motion resonances of the asteroid belt (which correspond to the known Kirkwood gaps) the secular Hamiltonian provides the integrable approximation for a Nekhoroshev like construction, whose small parameters include the eccentricities and inclinations of the planets (see Morbidelli and Guzzo 1997, Guzzo and Morbidelli 1997). Numerical implementations of this degeneracy removal have been done in Pavlović and Guzzo (2008), and in different contexts in Benettin et al. (1998), Schirinzi and Guzzo (2015) (see Section 5).

4. A SKETCH OF THE NEKHOROSHEV STABILITY MECHANISM

The stability argument. The different proofs of Nekhoroshev's theorem may be split essentially into three parts: a geometric part, devoted to the analysis of the resonances and the relevance of different small divisors in the different places of the action–space; an analytic part, devoted to the construction of the normal forms obtained by an averaging method; finally, a stability argument grants the confinement of the actions up to the exponentially long time. We here describe the stability mechanism which has been introduced in Nekhoroshev's 1977 paper, and then discussed and improved in several papers (Benettin et al. 1985, Pöschel 1993, Bounemoura and Marco 2011, Guzzo et al. 2014a, 2014b); for alternative proofs, based on the so-called simultaneous approximations, see Lochak (1992), Lochak and Neishtadt (1992), Bounemoura and Niederman (2012), Xue (2015).

Let us consider Hamiltonian (1):

$$H(I,\varphi) = h(I)+\varepsilon f(I,\varphi), \qquad (I,\varphi) \in B\times\mathbb{T}^n \ , \quad (8)$$

and the Fourier expansion of the perturbation:

$$f(I,\varphi) := \sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi} .$$

For any value of the perturbation parameter ε, a cut-off $K := K(\varepsilon) \sim 1/\varepsilon^a$ in the frequency space is defined, and the action space is covered by domains D_Λ labelled by all the (maximal) lattices Λ generated by j integer vectors $k_1, \cdots, k_j$, with $|k| \le K$. In each domain $D_\Lambda \times \mathbb{T}^n$ the Hamiltonian (8) is conjugate, via a close to the identity canonical averaging transformation, to a resonant normal form:

$$h(I) + \varepsilon g_\Lambda(I,\varphi) + \varepsilon r_\Lambda(I,\varphi) \qquad (9)$$

such that the resonant term g_Λ contains only Fourier harmonics in the resonant lattice Λ (i.e. with $k \in \Lambda$) and is ε–close to $\sum_{k\in\Lambda} f_k e^{ik\cdot\varphi}$; the remainder r_Λ is exponentially small with respect to K, and therefore with respect to $1/\varepsilon^a$. In the domain D_Λ all the harmonics related to some $k \in \Lambda$ cannot be normalized up to an exponentially small term, since the related divisor $|k\cdot\omega(I)|$ is small. Since the effect of the remainder on the dynamics can be large only on exponentially long times, at first we can describe the resonant dynamics by neglecting r_Λ in Hamiltonian (9):

$$H_\Lambda = h(I) + \varepsilon g_\Lambda(I,\varphi) \, .$$

In such approximation, $\dot I = -\varepsilon \frac{\partial g_\Lambda}{\partial\varphi}$ is a vector in the real space spanned by the resonant lattice Λ, and therefore the actions move in the linear space spanned by Λ through the initial value I_0: this is the so–called plane of fast-drift. The motions on the fast-drift plane can be complex, and are compatible with quite short Lyapunov times of order $T_{\rm L} \sim 1/\sqrt{\varepsilon}$ (we refer to the paper Benettin and Gallavotti (1985) for a description of fast drift motions). Without the steepness assumption on h, motions I_t with initial conditions in D_Λ could perform large variations in short times t smaller than $1/\varepsilon$, without leaving the fast drift plane of the resonant domain D_Λ. Steepness grants that the intersection of the fast drift planes with D_Λ is a set of small diameter. Therefore, the only possibility for I_t of performing large variations, is to visit different resonant domains: moving from one resonant domain to others, the actions could in principle move by quantities of order 1 in times much smaller than the exponentially long time. Again, steepness provides an obstruction to such possibility. More precisely, directly from its definition, one obtains that leaving a resonant domain D_Λ moving on a plane of fast drift, the small divisors increase with the power law ξ^{δ_j}, where ξ denotes the distance from the exact resonance. This means that the motion can leave the resonant domain D_Λ only to enter, in a short distance, in another resonant domain $D_{\Lambda'}$, with Λ' resonant lattice of dimension strictly smaller than the dimension of Λ.

Therefore, in a number of such hypothetical resonant passages smaller or equal to the dimension of Λ, the motion enters the non-resonant domain corresponding to the null resonant lattice, where the normal form is

$$h(I) + \varepsilon g_0(I) + \varepsilon r_0(I,\varphi). \qquad (10)$$

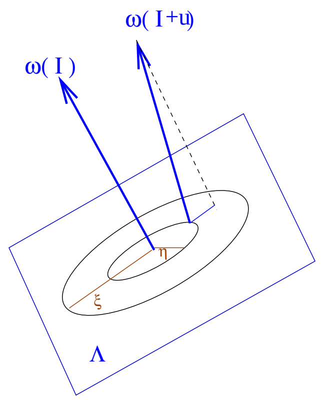

Fig. 1. *On geometric meaning of steepness. The action I is in multiple resonance with a set of linearly independent integer vectors, generating a lattice Λ: the frequency vector $\omega(I)$ is orthogonal to the linear space λ spanned by the lattice Λ; to leave the resonance along the plane of fast drift λ, arriving at distance ξ from the exact resonance, one has necessarily to increase $\|\pi_\lambda \omega\|$ by ξ^δ; the increment may be attained at an intermediate distance η.*

As a consequence, since the remainder r_0 is exponentially small, in the non-resonant region the motion is stopped up for an exponentially long time. Such an argument is called resonant trap, and is described in detail in Nekhoroshev (1977), and afterwards in Benettin et al. (1985), Guzzo et al. (2014a, 2014b).

The heuristic explanation given above is obtained by neglecting the exponentially small remainder r_Λ in all the normal forms. When r_Λ is considered, $\dot{I}$ is not exactly flattened on a fast-drift plane. Therefore, the exponentially small remainder can force a very slow change of the fast-drift plane, which is not controlled by steepness. The motion of the fast-drift plane can be relevant only on time scales longer than the Nekhoroshev stability time T_{N}, and is related to the so–called phenomenon of Arnold diffusion.

On the geometric meaning of steepness. We conclude this technical Section with a brief discussion about the geometric meaning of steepness. To understand Eq. (5) we have to think of its use in estimating the small divisors related to the different resonances. In the integrable approximation we have $\dot\varphi = \omega(I)$, and therefore, any resonant lattice Λ, defines a resonance in the action space through the equations:

$$k \cdot \omega(I) = 0 \quad \forall \ \ k \in \Lambda \backslash 0$$

or equivalently through the vector equation:

$$\pi_\lambda \omega(I) = 0,$$

where λ denotes the vector space spanned by the lattice Λ. In the action domain where $\|\pi_\lambda \omega(I)\|$ is small, even if not identically zero, the angle combinations $k\cdot\varphi$, $k \in \Lambda$, cannot be averaged. Therefore, since in the resonant domains the motions of the actions are flattened on the fast drift-planes, the resonant trap argument works if moving away from the exact resonance along the fast drift plane, i.e. by considering I such that $\pi_\lambda\omega(I) = 0$ and $u \in \lambda$, we have $\|\pi_\lambda \omega(I+u)\| \sim \|u\|^\delta$ with some positive δ. It is easy to prove that if h is quasi-convex we have $\delta = 1$ for any lattice Λ of dimension $j \in \{1, \ldots, n-1\}$. It would be therefore immediate to consider also functions with all possible values of $\delta \geq 1$. However, such generalization would really be poor and strongly not generic. Instead, the steepness requires that for any $u \in \lambda$ one has $\|\pi_\lambda\omega(I+u')\| \sim \|u\|^\delta$ for some $u' \in \lambda$, $0 < \|u'\| \leq \|u\|$ (see Fig. 1). Therefore, to leave the resonance along the plane of fast drift, arriving at distance ξ from the exact resonance, one has necessarily to increase $\|\pi_\lambda\omega\|$ by ξ^δ at some intermediate point. The exponent δ, called steepness index, is characteristic of the dimension of the resonant lattice, so that we have $\delta_1, \ldots, \delta_{n-1}$ steepness indices.

5. SOME APPLICATIONS OF THE THEOREM

The most important outcome of Nekhoroshev's theorem to Solar System dynamical investigations has been that of providing a new way of thinking to its long–term stability. It is remarkable that, in addition to the specific systems which have been claimed to be stable in the long–term due to a direct application of a statement of Nekhoroshev's theorem, many numerical investigations have been interpreted by referring to the description of the dynamics obtained from the resonant normal forms considered in the proof of the theorem (see Section 4). Below, we quote some of the applications which have been performed up to now.

The Planetary System. Already in the paper Nekhoroshev (1977) we find the idea of using the theorem about the exponential stability of quasi–integrable systems to prove the long-term stability of planetary systems (Nekhoroshev 1977, Section 12; see also Niederman 1996). Nekhoroshev's idea was to average out from the Hamiltonian the mean anomalies of the planets up to an exponentially small remainder, in a domain which excludes the low order mean motion resonances. The average of the mean anomalies implies that the values of the semi-major axes are stable up to exponentially long-times, provided that meanwhile the values of eccentricities and inclinations remain small. The variations of the eccentricities and inclinations have been constrained by using the fact that, in the averaged system, the configuration corresponding to all the planets in circular and co-planar orbits is a maximum of the angular momentum. A stronger stability result valid for the planar three–body problem, and a discussion of the steepness of planetary systems, can be found in Pinzari (2013).

A different approach has been developed by a group of authors to prove the long–term stability of a restricted model for the outer planets of our Solar System for the actual values of the masses of the planets. Precisely, by using a combination of both the KAM and the Nekhoroshev theorems (see Morbidelli and Giorgilli 1995), the authors proved that the motion of some restricted planetary problems (for example the Sun-Jupiter-Saturn system or the planar Sun-Jupiter-Saturn-Uranus system) remain close to an invariant torus of the system for times comparable to Solar System lifetime (see Giorgilli et al. 2009, Sansottera et al. 2013). The result is obtained also with the help of computer algebra implementation of averaging transformations.

The Trojan asteroids. One of the first applications of Nekhoroshev's theorem to Solar System dynamics was motivated by the need of proving stability of the Trojan asteroids (Giorgilli et al. 1989, Giorgilli and Skokos 1997 , Skokos and Dokoumetzidis 2001, Efthymiopoulos and Sándor 2005, Lhotka et al. 2008). Since the Trojan asteroids librate around the Sun-Jupiter Lagrangian point L4, the use of Nekhoroshev's theorem in this context is motivated by the fact that near an elliptic equilibrium point, such as the equilibria L4 and L5 of the restricted circular three–body problem, the Birkhoff normal forms of the system are quasi-integrable, and the small parameter is represented by the distance from the equilibrium. The original idea, then developed with models of increasing complexity, was to implement with the help of a computer algebra the average of the Hamiltonian around the Lagrangian equilibrium point L4 by performing the largest number of Birkhoff normalization steps which are allowed by the non–resonant properties of the frequencies of small oscillations around L4 in the Sun–Jupiter system. The authors therefore construct a non resonant normal form Hamiltonian, with a direct estimate of the remainder; the inverse of the norm of the gradient of the remainder has been used as an estimate of the Nekhoroshev stability time. Of course, since the small parameter is represented by the distance from the equilibrium, the stability time converges to infinity at L4, and it is interesting to determine the region around L4 where stability time is larger than the age of the Solar System.

Lagrangian equilibria L4-L5. The long-term stability properties of the Lagrangian equilibrium points L4-L5 strongly depends on value of the reduced mass μ defining the three-body problem. In fact, frequencies of small oscillations around L4-L5 depend on μ, and very different resonances are encountered for different values of μ. For example, by considering the spatial circular restricted three–body problem when μ corresponds to the Sun–Jupiter system the equilibrium is far from strong resonances, while other values of μ correspond to very low order resonances. The Nekhoroshev stability of elliptic equilibria has been conjectured in the paper Nekhoroshev (1977), and has been proved in Fassò et al. (1998), Guzzo et al. (1998), Niederman (1998), without any Diophantine condition on frequencies, and under the assumption of quasi–convexity, or directional quasi-convexity (Fassò et al. 1998) of the Birkhoff normal form of order four. In Benettin et al. (1998) it was shown that, in the spatial circular restricted three–body problem, the equilibria L4 and L5 are Nekhoroshev stable, in the sense that the stability time in a neighbourhood of radius d from the equilibrium increases exponentially with a power of $1/d$, for all values of the reduced mass up to the Routh critical values, except for three values of μ, which we denote μ_1, μ_2 and μ_3. For μ_1 and μ_2 the Nekhoroshev result is not valid since, due to the presence of low order resonances at μ_1 and μ_2, the Birkhoff normal forms cannot be constructed. For all $\mu \neq \mu_1$, μ_2 and μ_3 the Birkhoff normal forms could be constructed, and were quasi–convex, directionally quasi–convex, or three jet non degenerate (see condition Eq. (7)). The steepness at μ_3 has been shown recently (Schirinzi and Guzzo 2015).

Spin-orbit rotations. The rotation of many celestial bodies can be represented as a perturbation of the so–called Euler–Poinsot rigid body. For example, the D'Alembert model of planetary precession consisting of an oblate rigid body orbiting about a central star in a Keplerian motion, is one of the basic models for studying the spin–orbit problem. The applications of Nekhoroshev's theorem to perturbations of the Euler–Poinsot rigid body have been developed in Benettin and Fassó (1996), Benettin et al. (1997, 2004). The proper degeneracy of the Euler–Poinsot Hamiltonian, which does not depend on one of the action variables determining the orientation in space of the angular momentum M of the body, leaves opened the possibility of large scale chaotic motions for the orientation of M in space, while its norm $\|M\|$, and its projection on a symmetry axis, remain almost constant up to times which increase exponentially with $1/\sqrt{\Omega}$, Ω denoting the norm of the angular velocity. We remark that the coexistence of action variables which are stable up to very long-times with action variables which, in the same time interval, can perform large scale chaotic motions, is typical of degenerate systems, and in particular of the rigid body dynamics.

The application of these results to the spin–orbit problem requires specific modifications, to take into account the motion of the center of mass of the body, and the specific expansion of the perturbation. Some applications of KAM and Nekhoroshev theory to the D'Alembert model of planetary precession have been done in Chierchia and Gallavotti (1994), Biasco and Chierchia (2002), under the hypothesis of small eccentricity of the orbit. In Benettin et al. (2008) the model has been studied without any restriction on the orbital eccentricity. Using a combination of the Nekhoroshev's theorem and of the theory of adiabatic invariants, the authors found that, as soon as the eccentricity is not very small, and the angular velocity is far from the minimum (maximum) principal axis of inertia, large scale chaotic motions are possible on relatively short time scales, an instability very similar to the ones discussed in Benettin and Fassó (1996), Benettin et al. (1997, 2004, 2006).

The Main Belt Asteroids. The distribution of asteroids between Mars and Jupiter is sculpted by the mean motion and secular resonances. For example, the Kirkwood gaps, the asteroid families, the three–body resonances, and the secular resonances represent very different dynamics with very different related stability times. In Morbidelli and Guzzo (1997), Guzzo and Morbidelli (1997), it was proposed to use Nekhoroshev's theorem to understand and classify the broad spectrum of different dynamics which are observed in the Main Belt. This project needs to adapt the theorem to local properties of resonances and perturbations, which change significantly in different parts of the belt. First, one needs to split the Hamiltonian in a quasi–integrable form. To distinguish all the complex dynamical behaviours of the belt, it was suggested to perform preliminarly a finite number of averaging transformations which reduce the Hamiltonian of the asteroid problem $H_{\rm Ast}$ to some resonant non degenerate normal form $H^*_{\rm Ast}$, like Hamiltonian (4), see Morbidelli and Guzzo (1997), Guzzo and Morbidelli (1997). As well–known, the construction of normalizing transformations is strictly related to the specific expansion of the perturbation on a specific domain. For example, in Guzzo and Morbidelli (1997) it was shown the the averaged Hamiltonian $H^*_{\rm Ast}$ is in quasi–integrable form also in the mean motion resonances with the planets, except for a small number of low order resonances which include the Kirkwood gaps. Second, one needs to compute the steepness of the integrable approximation of $H^*_{\rm Ast}$. To do this, one needs to represent the so–called Kozai Hamiltonian with action-angle variables, to compute its derivatives up to order three, and then to check if the Hamiltonian is three–jet non degenerate, or quasi–convex. This procedure has been described and numerically implemented for the members of the Koronis and Veritas families in Pavlović and Guzzo (2008). Finally, one has to check that the existent small parameters of $H^*_{\rm Ast}$ are smaller than the threshold of application of Nekhoroshev's theorem and to compute the stability time $T_{\rm N}$. This part has not been done yet. A different approach, followed in Guzzo et al. (2002a), consists in checking if the numerically integrated dynamics of individual asteroids are compatible with the regime of validity of Nekhoroshev's theorem. In fact, as it has been proved in Guzzo and Benettin (2001) (see also Guzzo et al. 2002b), the Fourier spectrum of solutions of systems which are in a regime of validity of Nekhoroshev's theorem is very peculiar. The peculiarity can easily be checked even with short-term numerical computations, of the order of the Lyapunov times for chaotic motions. The technique has been successfully applied to individual asteroids in chaotic motion.

6. CONCLUSIONS

The Nekhoroshev theorem provides an effective stability time for a given system which is much larger than its Lyapunov time. The dependence of the Nekhoroshev stability time on a set of physical parameters (the small parameters) and of phase–space parameters (the steepness indices), as well as on the possibility of performing local preliminary averaging transformations, is necessery importation which must be used to represent the complexity of the stability properties of complicated systems defined by the Solar System dynamics. The dynamical complexity of the asteroid belt and the increasing number of asteroid discoveries make the main belt an ideal field for future applications of the theorem.

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REFERENCES

Arnold, V. I.: 1963, *Russ. Math. Surv.*, **18**, 9.

Arnold, V. I.: 1964, *Sov. Math. Dokl.*, **6**, 581.

Benettin, G. and Fassò, F.: 1996, *Nonlinearity*, **9**, 137.

Benettin, G., Fassò, F. and Guzzo, M.: 1997, *Nonlinearity*, **10**, 1695.

Benettin, G., Fassò, F. and Guzzo, M.: 2004, *Comm. Math. Phys.*, **250**, 133.

Benettin, G., Fassò, F. and Guzzo, M.: 2006, *Regul. Chaotic Dyn.*, **11**, 1, 1.

Benettin, G., Fassò, F. and Guzzo M.: 1998, *Regul. Chaotic Dyn.*, **3**, 56.

Benettin, G., Galgani, L. and Giorgilli, A.: 1985, *Celest. Mech.*, **37**, 1.

Benettin, G. and Gallavotti, G.: 1985, *J. Stat. Phys.*, **44**, 293.

Benettin, G., Guzzo, M. and Marini, V.: 2008, *Celest. Mech. Dyn. Astron.*, **101**, 203.

Biasco L., Chierchia L.: 2002, *Celest. Mech. Dyn. Astr.*, **83**, 223.

Bounemoura, A. and Marco, J. P.: 2011, *Nonlinearity*, **24**, 97.

Bounemoura, A. and Niederman, L.: 2012, *Annales de l'Institut Fourier*, **62**, 277.

Celletti, A. and Chierchia, L.: 1995, A Constructive Theory of Lagrangian Tori and Computer-assisted Applications, in Dynamics Reported, New Series, Vol. 4, Springer, Berlin.

Celletti, A. and Chierchia, L.: 1997, *Comm. Math. Phys.*, **186**, 413.

Celletti A. and Ferrara L.: 1996, *Celest. Mech. Dyn. Astron.*, **64**, 261.

Cincotta, P. M., Efthymiopoulos, C., Giordano, C. M. and Mestre, M. F.: 2014, *Physica D*, **266**, 49.

Chierchia, L. and Gallavotti, G.: 1994, *Ann. Inst. Henri Poincaré, Phys. Théor.*, **60**, 1 (*Erratum*, 1998, *Ann. Inst. Henri Poincaré, Phys. Théor.*, **68**, 135).

Chirikov, B. V.: 1971, Preprint N267, Institute of Nuclear Physics, Novosibirsk (1969) Engl. Trans., CERN Trans. 71-40.

Efthymiopoulos, C.: 2008, *Celest. Mech. Dyn. Astron.*, **102**, 49.

Efthymiopoulos, C. and Sándor, Z.: 2005, *Mon. Not. R. Astron. Soc.*, **364**, 253.

Efthymiopoulos, C., and Harsoula, M.: 2013, *Physica D*, **251**, 19.

Efthymiopoulos, C., Voglis, N. and Contopoulos, G.: 1998, in "Analysis and Modelling of discrete dynamical systems", eds. D. Benest and C. Froeschlé, Gordon and Breach Science Publishers, Philadelphia.

Fassò, F., Guzzo, M. and Benettin G.: 1998, *Comm. Math. Phys.*, **197**, 347.

Froeschlé, Cl., Guzzo, M. and Lega, E.: 2005, *Celest. Mech. Dyn. Astron.*, **92**, 243.

Giorgilli, A., Delshams, A., Fontich, E., Galgani, L. and Simó, C.: 1989, *J. Diff. Eq.*, **77**, 167.

Giorgilli, A., Locatelli, U. and Sansottera, M.: 2009, *Celest. Mech. Dyn. Astron.*, 104, 159.

Giorgilli, A. and Skokos, C.: 1997, *Astron. Astrophys.*, **317**, 254.

Guzzo, M.: 1999, *Regular and Chaotic Dynamics*, **4**, 2.

Guzzo, M.: 2005, *Icarus*, **174**, 273.

Guzzo, M.: 2006, *Icarus*, **181**, 475.

Guzzo, M. and Benettin, G.: 2001, *Discrete and Continuous Dynamical Systems-Series B*, **1**, 1.

Guzzo, M., Chierchia, L. and Benettin, G.: 2014a, *Rend. Lincei Mat. Appl.*, **25**, 293.

Guzzo, M., Chierchia, L. and Benettin, G.: 2014b, arXiv:1403.6776

Guzzo, M., Fassò, F. and Benettin, G.: 1998, *Mathematical Physics Electronic Journal*, **4**, 1.

Guzzo, M., Knežević, Z. and Milani, A.: 2002a, *Celest. Mech. Dyn. Astron.*, **83**, 121.

Guzzo, M., Lega, E. and Froeschlé, Cl.: 2002b, *Physica D*, **163**, 1.

Guzzo, M., Lega, E. and Froeschlé, Cl.: 2005, *Discrete and Continuous Dynamical Systems B*, **5**, 687.

Guzzo, M., Lega, E. and Froeschlé, Cl.: 2006, *Nonlinearity*, **19**, 1049.

Guzzo, M., Lega, E. and Froeschlé, Cl.: 2011, *Chaos*, **21**, 3.

Guzzo, M. and Morbidelli, A.: 1997, *Celest. Mech. Dyn. Astron.*, **66**, 255.

Hénon, M. and Heiles, C.: 1964, *Astron. J.*, **69**, 73.

Kolmogorov, A. N.: 1954, *Dokl. Akad. Nauk. SSSR*, **98**, 527.

Konishi, T. and Kaneko, K.: 1990, *J. Phys. A: Math. Gen.*, **23**, L715.

Laskar, J.: 1990, *Icarus*, **88**, 266.

Laskar, J.: 1993, *Physica D*, **67**, 257.

Laskar, J. and Robutel, P.: 1993, *Nature*, **361**, 608.

Laskar, J.: 2008, *Icarus*, **196**, 1.

Lega, E., Guzzo, M. and Froeschlé, Cl.: 2003, *Physica D*, **182**, 179.

Lichtenberg, A. and Aswani, M. A.: 1998, Arnold diffusion in many weakly coupled mappings. *Phys. Rev. E*, **57**, 5325.

Lochak, P.: 1992, *Russ. Math. Surv.*, **47**, 57.

Lochak, P. and Neishtadt, A.: 1992. *Chaos*, **2**, 495.

Lhotka, Ch., Efthymiopoulos, C. and Dvorak, R.: 2008, *Mon. Not. R. Astron. Soc.*, **384**, 1165.

Milani, A. and Nobili A.: 1992, *Nature*, **357**, 569.

Morbidelli, A. and Giorgilli, A.: 1995, *J. Stat. Phys.*, **78**, 1607.

Morbidelli, A. and Guzzo, M.: 1997, *Celest. Mech. Dyn. Astron.*, **65**, 107.

Moser, J.: 1958, *Comm. on Pure and Appl. Math.*, **11**, 81.

Murray, N. and Holman, M.: 1999, *Science*, **283**, 1877.

Murray, N., Holman, M. and Potter, M.: 1998, *Astron. J.*, **116**, 2583.

Nekhoroshev, N. N.: 1973, *Mat. Sb.*, **90**, 432.

Nekhoroshev, N. N.: 1977, *Russ. Math. Surv.*, **32**, 1.

Nekhoroshev, N. N: 1979, *Tr. Semin. Petrovsk.*, **5**, 5 (in Oleinik, O.A. (ed.) Topics in Modern Mathematics, Petrovskii Semin., No. 5. Consultant Bureau, New York, 1985).

Neishtadt, A. I.: 1987, *Sov. Phys. Dokl.*, **32**, 571.

Nesvorny, D. and Morbidelli, A.: 1998, *Astron. J.*, **116**, 3029.

Niederman, L.: 1996, *Nonlinearity*, **9**, 1703.

Niederman, L.: 1998, *Nonlinearity*, **11**, 1465.

Pavlović, R. and Guzzo, M.: 2008, *Mon. Not. R. Astron. Soc.*, **384**, 1575.

Pinzari, G.: 2013, *Regular and Chaotic Dynamics*, **18**, 860.

Pöschel, J.: 1993, *Math. Z.*, **213**, 187.

Sansottera, M., Locatelli, U. and Giorgilli, A.: 2013, *Mathematics and Computers in Simulation*, **88**, 1.

Schirinzi, G. and Guzzo, M.: 2013, *J. Math. Phys.*, **54**, 072702.

Schirinzi, G. and Guzzo, M.: 2015, *Regular and Chaotic Dynamics*, **20**, 1.

Skokos, C. and Dokoumetzidis, A.: 2001, *Astron. Astrophys.*, **367**, 729,

Sussman, G. J. and Wisdom, J.: 1988, *Science*, **241**, 433.

Sussman, G. J. and Wisdom, J.: 1992, *Science*, **257**, 56.

Todorović, N., Guzzo, M., Lega, E. and Froeschlè, Cl.: 2011, *Celest. Mech. Dyn. Astron.*, **110**, 389.

Wayne, B. H., Malykh, A. V. and Danforth, C. M.: 2010, *Mon. Not. R. Astron. Soc.*, **407**, 1859.

Wisdom, J.: 1982, *Astron. J.*, **87**, 577.

Wisdom, J. and Peale, S. J.: 1984, *Icarus*, **58**, 137.

Wood B. P., Lichtenberg A. and Lieberman M. A.: 1990, *Phys. Rev. A*, **42**, 5885.

Xue, J.: 2015, *Discrete and continuous dynamical systems*, **35**, 3827.

ТЕОРЕМА НЕХОРОШЕВА И ДУГОРОЧНА СТАБИЛНОСТ У СУНЧЕВОМ СИСТЕМУ

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Теорема Нехорошева је последњих деценија често навођена као референтна теорема за објашњавање динамике неколико система који су стабилни у дугим временским интервалима. Динамика Сунчевог система даје широк спектар могућих и корисних примена. Заправо, без обзира на компликоване моделе који су коришћени да се нумерички интеграле једначине које описују реалистичну динамику Сунчевог система што је прецизније могуће, када су добијена решења хаотична, поузданост нумеричке интеграције је ограничена и потребна је теоријска анализа дугорочне стабилности. Након прве формулације теореме Нехорошева 1977. године постигнута су многа побољшања. Са једне стране, алтернативни докази саме теореме су довели до сталног напретка у проценама стабилности; са друге стране, развијени су додаци који су неопходни за примену теореме на системе од интереса у динамици Сунчевог система, конкретно они који се тичу уклањања дегенерисаности и имплементације компјутерски добијених доказа. У овом прегледном раду разматрају се мотиви и резултати који су теорему Нехорошева учинили референтним резултатом при разматрању стабилности у бројним применама на динамику Сунчевог система.