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UNIVERSAL SYMBOLIC EXPRESSION FOR RADIAL DISTANCE OF CONIC MOTION

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SUMMARY: In the present paper, a universal symbolic expression for radial distance of conic motion in recursive power series form is developed. The importance of this analytical power series representation is that it is *invariant* under many operations because the result of addition, multiplication, exponentiation, integration, differentiation, etc. of a power series is also a power series. This is the fact that provides excellent flexibility in dealing with analytical, as well as computational developments of problems related to radial distance. For computational developments, a full recursive algorithm is developed for the series coefficients. An efficient method using the continued fraction theory is provided for series evolution, and two devices are proposed to secure the convergence when the time interval $(t - t_0)$ is large. In addition, the algorithm does not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits. Numerical applications of the algorithm are given for three orbits of different eccentricities; the results showed that it is accurate for any conic motion.

Key words. celestial mechanics

1. INTRODUCTION

It is undoubtedly true that the analytical formulae of space dynamics usually offer much deeper insight into the nature of problems they refer to. Moreover, the nowadays existing symbols used for manipulating digital computer programs opened the gate towards establishing a new branch of space dynamics known as the algorithmization of space dynamics (Brumberg 1995). A great effort has been devoted up to now, and is being devoted at present to develop symbolic computing algorithms for some problems of astrodynamics as well as astrophysics (e.g. Sharaf and Saad 1997, Sharaf et al. 1998, Sharaf 2005, Sharaf 2008, Sharaf and Sendi 2011, Sharaf et al. 2012, Sharaf and Saad 2013).

In the absence of closed analytical solution of a given differential system, a power series solution (which is of course assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is much more convenient than the elementary functions especially when the problems are to be studied on computers. In fact, most computers often use series in calculations of the majority of elementary functions.

Coping with the above important line of recent approach, the present paper is devoted to establish a universal symbolic expression for radial distance of conic motion.

Radial distances are vital to a class of orbit determination problems which depend on range measurements (Vallado 1997). The importance of these analytical power series representation is that they are invariant under many operations because addition, multiplication, exponentiation, integration, differentiation, etc of a power series is also a power series. This is the fact which provides excellent flexibility in dealing with analytical, as well as computational developments of problems related to radial distance. For computational developments, a full recursive algorithm is developed for the series coefficients. An efficient method using the continued fraction theory is provided for series evolution, and two devices are proposed to secure the convergence when the time interval $(t - t_0)$ is large. In addition, we do not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.

The importance of the universal recurrent power series representation of the radial distance established in the present paper are due to the following facts. Its universal nature avoids critical situation in some orbital systems; this is because the type of an orbit is occasionally changed by perturbing forces during finite interval of time. Thus far, we have been obliged to use different functional representations for motion depending on energy state (elliptic, parabolic, or hyperbolic), and the simulation code must then contain branching to handle a switch from one state to another. In cases when this switching is not smooth, branching can occur many times during a single integration time-step causing some numerical problems. Consequently, universal formulations are desperately needed, so that the orbit determination is free of complications, since a single functional representation suffices to describe all possible states. In addition, the recursive nature of the developed power series (recurrent power series) facilitates their computations. By these points mentioned above, we claim that our method may be optimal in terms of the orbit determination problem.

2. BASIC FORMULATION

2.1. Differential equations

The polar equation for relative motion of the two body problem is given as

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2, \tag{1}$$

where r is the radial distance, θ the true anomaly, μ is the gravitational parameter, and $\dot{\theta} = \sqrt{\mu p}/r^2$, where p is the orbital parameter. Here a dot over a symbol denotes its time derivative. Since p is constant for the two body problem, Eq. (1) could be written as

$$\ddot{q} = -\varepsilon q, \tag{2}$$

where

$$q = r - p,\ \varepsilon = \mu/r^3. \tag{3}$$

2.2. Lagrange's Fundamental Invariants

Lagrange's fundamental invariants (Battin 1999) ε, λ and ψ are defined as

$$\varepsilon = \mu/r^3. \tag{4}$$

$$\lambda = \frac{1}{r^2} < \vec{r}, \vec{v} >, \tag{5}$$

$$\psi = \frac{1}{r^2} < \vec{v}, \vec{v} >, \tag{6}$$

where $< \vec{A}, \vec{B} >$ denotes the scalar product of vectors $\vec{A}$ and $\vec{B}$. The quantities ε, λ and ψ are "invariant" because they are independent of the selected coordinate system and "fundamental" because they form a closed set under the operation of time derivative, where

$$\frac{d\epsilon}{dt} = -3\varepsilon\lambda, \tag{7}$$

$$\frac{d\lambda}{dt} = \psi - \varepsilon - 2\lambda^2, \tag{8}$$

$$\frac{d\psi}{dt} = -2\lambda(\varepsilon + \psi). \tag{9}$$

3. POWER SERIES SOLUTION

3.1. The basic differential equations

The basic differential equations that concern us in the subsequent analysis are Eqs. (2) and (7) – (9) written as

$$\frac{d^2q}{dt^2} + \varepsilon q = 0, \tag{10}$$

$$\frac{d\varepsilon}{dt} + 3\varepsilon\lambda = 0, \tag{11}$$

$$\frac{d\psi}{dt} + 2\lambda(\varepsilon + \psi) = 0, \tag{12}$$

$$\frac{d\lambda}{dt} + \varepsilon + 2\lambda^2 - \psi = 0, \tag{13}$$

where ε, λ and ψ are defined by Eqs. (4) – (6).

3.2. Power series solutions

Power series solutions for the above set of differential equations could be developed as follows from Battin (1999). Expanding each of the functions q, ε, λ and ψ in a Taylors series in time we have

$$q = \sum_{n=0}^{\infty} q_n (t - t_0)^n, \tag{14}$$

$$\varepsilon = \sum_{n=0}^{\infty} \varepsilon_n (t - t_0)^n, \tag{15}$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n (t - t_0)^n, \tag{16}$$

$$\psi = \sum_{n=0}^{\infty} \psi_n (t - t_0)^n. \tag{17}$$

The procedure is to substitute the four series given by Eqs. (14) − (17) into the four differential Eqs. (10) − (13) and then solve for the coefficients $q_n, \varepsilon_n, \lambda_n$ and ψ_n by comparison of the coefficients of powers of time. The central mathematical device used is the general relation

$$\left(\sum_{n=0}^{\infty} \alpha_n X^n\right)\left(\sum_{n=0}^{\infty} \beta_n X^n\right) = \sum_{n=0}^{\infty}\sum_{\nu=0}^{n} \alpha_\nu \beta_{n-\nu} X^n, \tag{18}$$

which converts the product of two infinite series to a double summation.

3.3. Recurrence relations

The resulting recurrence relations are

$$q_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{i=0}^{n} \epsilon_i q_{n-i}, \tag{19}$$

$$\varepsilon_{n+1} = \frac{-3}{(n+1)} \sum_{i=0}^{n} \epsilon_i \lambda_{n-i}, \tag{20}$$

$$\lambda_{n+1} = \frac{1}{(n+1)} \left\{ \psi_n - \varepsilon_n - 2\sum_{i=0}^{n} \lambda_i \lambda_{n-i} \right\}, \tag{21}$$

$$\psi_{n+1} = \frac{-2}{(n+1)} \sum_{i=0}^{n} \lambda_i (\varepsilon_{n-i} + \psi_{n-i}). \tag{22}$$

3.4. The starting values

The starting values for the recurrence relations of Eqs. (19)-(22) are $q_0 \equiv q(t_0), q_1 \equiv q(\dot{t_0}), \varepsilon_0 \equiv \varepsilon(t_0), \lambda_0 \equiv \lambda(t_0)$ and $\psi_0 \equiv \psi(t_0)$ could be obtained from the known position and velocity vectors $\vec{r}_0(x_0, y_0, z_0)$ and $\dot{\vec{r}}_0(\dot{x_0}, \dot{y_0}, \dot{z_0})$ at time t_0 by the following algorithm

1. $r_0 = \left(x_0^2 + y_0^2 + z_0^2\right)^{1/2}$;
2. $\varepsilon_0 = \mu / r_0^3$;
3. $q_1 = (x_0 \dot{x_0} + y_0 \dot{y_0} + z_0 \dot{z_0})/r_0$;
4. $\lambda_0 = q_1 / r_0$;
5. $\psi_0 = (\dot{x_0}^2 + \dot{y_0}^2 + \dot{z_0}^2)/r_0^2$;
6. $h_x = y_0 \dot{z_0} - z_0 \dot{y_0}$;
7. $h_y = z_0 \dot{x_0} - x_0 \dot{z_0}$;
8. $h_z = x_0 \dot{y_0} - y_0 \dot{x_0}$;
9. $q_0 = r_0 - (h_x^2 + h_y^2 + h_z^2)/\mu$;

Having obtained the q's, ε's, λ's and ψ's coefficients recursively from Eqs. (19)-(22), the power series expansion of r

$$r = r_0 + \sum_{n=1}^{\infty} q_n (t - t_0)^n, \tag{23}$$

is valid for any conic orbit (elliptic, parabolic, hyperbolic).

4. SYMBOLIC AND NUMERICAL APPLICATIONS

4.1. Symbolic expansions

Using the symbolic manipulation capability of the software package *Mathematica*, we generate the coefficients $q_j; j = 2, 3, ..., 10$ in terms of the known initial values $q_0, q_1, \varepsilon_0, \lambda_0$ and ψ_0, which are listed in Appendix A.

4.2. Numerical applications

The numerical evaluation of power series (say) q:

$$q = \sum_{n=0}^{\infty} q_n (t - t_0)^n, \tag{24}$$

may diverge when $\triangle = (t - t_0)$ is large. To avoid this difficulty, the following two devices could be used.

4.2.1 Canonical units

We use the following transformation rules (Vallado 1997)

Rules 1: Physical to Canonical

$\mu = 398600.4415$ km^3/s^2 $\rightarrow \mu = 1 ER^2/TU^3$

Distance p in km $\rightarrow$ p^* in ER such that $p^* = p/ER$

time t in s $\rightarrow$ time t^* in TU such that $t^* = t/f_1$

speed s in km/s $\rightarrow$ speed s^* in ER/TU such that $s^* = s/f_2$ where ER=6378.1363 km is the mean equatorial radius of the Earth, $f_1 = 806.8109$ s and $f_2 = 7.90536$ km/s.

After performing the computations using these canonical units, we can convert the results into the physical units by applying the following

Rules 2: Canonical to Physical

$\mu = 1ER^2/TU^3 \rightarrow \mu = 398600.4415\ \mathrm{km}^3/\mathrm{s}^2$

Distance p^* in ER $\rightarrow$ Distance p in km such that $p = p^* \times ER$

Time t^* in TU $\rightarrow$ Time t in s such that $t = t^* \times f_1$

speed s^* in ER/TU $\rightarrow$ Speed s in km/s such that $s = s^* \times f_2$.

If the time interval $\triangle$ were still large (because in orbit determination $\triangle$ is usually small, but we include the device to cover all possibilities that may occurr) we can use the following device.

4.2.2. Division of the time interval

In the case in which time interval is large we can use the process of repeat decrementing of the time interval several times, so each time interval could be made as small as we desired.

4.2.3. Continued fraction

In fact, continued fraction expansions are generally far more efficient tool for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series. Due to the importance of accurate evaluations and the efficiency of continued fractions, we propose to use them as the computational tools for evaluating the radial distance. To do so, two steps are to be performed:
1. Transform the given power series into continued fraction (point a)
2. Evaluating the resulting continued fraction (point b)

(a) Euler's transformation

Generally an infinite series (a power series is its special case) of functions could be converted into a continued fraction according to Euler's transformation (Battin 1999) which is

$$\sum_{k=0}^{\infty} U_k \equiv \frac{n_1}{d_1 + \frac{n_2}{d_2 + \frac{n_3}{d_3 + \frac{n_4}{\cdots}}}} \equiv \frac{n_1}{d_1+}\frac{n_2}{d_2+}\frac{n_3}{d_3+}\frac{n_4}{d_4+} + \ldots \quad (25)$$

where

$$n_1 = U_0; n_2 = U_1; n_i = -U_{i-1} \times U_{i-3}, \forall i \geq 3, \quad (26)$$

$$d_1 = 1; d_j = U_{j-2} + U_{j-1}, \forall i \geq 2. \quad (27)$$

(b) Top-down continued fraction evaluation

There are several methods available for evaluation of a continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n-th convergent were accumulated separately with three-term recurrence formulae. The drawback of the former method, obviously, is that one has to descend far down the fraction to ensure convergence. The drawback of the latter method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi (1967) proposed a very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as

$$q = \frac{n_1}{d_1+}\frac{n_2}{d_2+}\frac{n_3}{d_3+}\ldots \quad (28)$$

then initialize the following parameters

$$a_1 = 1; b_1 = n_1/d_1, q_1 = n_1/d_1, \quad (29)$$

and iterate $(k = 1, 2, \ldots)$ according to

$$a_{k+1} = \frac{1}{1 + \frac{n_{k+1}}{d_k d_{k+1}} a_k}, \quad (30)$$

$$b_{k+1} = (a_k - 1)b_k; q_{k+1} = q_k + b_{k+1}. \quad (31)$$

In the limit, the q sequence converges to the value of the continued fraction.

4.2.4. Numerical examples

In what follows, we shall consider three orbits, the first is elliptic, the second is parabolic, while the third is hyperbolic. The initial position and velocity vectors of the orbits are listed in Tables 1 and 2.

Table 1. The initial position vector.

Orbit	x_0(km)	y_0(km)	z_0(km)
1	5096.530625	3997.328251	-1767.35171
2	-1616.940994	7756.699643	-7712.188395
3	10000	0.0	0.0

Table 2. The initial velocity vector.

Orbit	$\dot{x_0}$(km/s)	$\dot{y_0}$(km/s)	$\dot{z_0}$(km/s)
1	4.683016085	0.602386847	4.217758697
2	-0.6730303137	8.434930957	0.7055483746
3	0.0	0.0	9.2

Table 3. The values of the q's coefficients and $\triangle^j$ for the three orbits.

j	q_{E_j}	q_{p_j}	q_{H_j}	$\triangle^j$
1	0.354604	0.698714	0.0	0.619724
2	-0.206308	0.0255706	0.228516	0.384058
3	0.0188294	-0.0326577	0.0	0.23801
4	-0.0032861	0.0170693	-0.0215942	0.1475
5	-0.0036932	-0.00697602	0.0	0.0914095
6	0.00443825	0.00219	0.0036222	0.0566486
7	-0.00398278	-0.000350869	0.0	0.0351065
8	0.00313643	-0.000167967	-0.00074509	0.0217563
9	-0.0023212	0.00020834	0.0	0.0134829
10	0.00164359	-0.000131368	0.000170544	0.00835569

For all three orbits we take $t_0 = 0, t = 500$ s, $m = 10$, where m is the number of terms in the series of Eq. (23). Applying the recurrent computations of Section 3 together with Rules 1 of Subsection 4.2.1 we get respectively for the value of q_0 of the three orbits the values $q_{E_0} = 0.481274$ ER, $q_{p_0} = -0.266446$ ER, and $q_{H_0} = 0.481274$ ER. The other coefficients $q_j; j = 1, 2, ..., m$ are listed in the first three columns of Table 3 together with the value $\triangle^j$ in the fourth column. Finally we compare the value of the radial distance r at $t = 500\,\text{s} \equiv 0.619724$ TU as computed from the above algorithm with its exact value; these comparisons are listed in Table 4 for the three orbits.

Table 4. Comparison between the computed and exact values of r.

Orbit	Computed r(ER)	Exact r(ER)
1	1.1969909	1.1969867
2	2.17063610	2.17063456
3	1.6526248	1.6526326

Although we used relatively small number of terms for the power series, Table 4 shows that the present algorithm is accurate enough ($\approx 10^{-5}$) for predicting radial distance of any conic orbit.

In concluding the present paper, we stress that universal symbolic expression for radial distance of conic motion in recursive power series forms is developed. The importance of this analytical power series representation is that they are invariant under many operations because, the addition, multiplication, exponentiation, integration, differentiation, etc. of a power series is also a power series. This is the fact which provides excellent flexibility in dealing with analytical as well as computational developments of problems related to radial distance. For computational developments, a full recursive algorithm is developed for coefficients of the series. Also an efficient method using the continued fraction theory is provided for series evolution and, moreover, two devices are proposed to secure the convergence when the time interval $(t - t_0)$ is large. In addition, we do not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.

Numerical applications of the algorithm are given for three orbits of different eccentricities; the results showed that it is accurate for any conic motion.

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Appendix A: SYMBOLIC EXPRESSIONS OF THE Q'S COEFFICIENTS

$q_2 = -q_0\varepsilon_0/2!,$

$q_3 = -\varepsilon_0(q_1 - 3q_0\lambda_0)/3!,$

$q_4 = \varepsilon_0(6q_1\lambda_0 + q_0(-2\varepsilon_0 + 3(-5\lambda_0^2 + \psi_0)))/4!,$

$q_5 = \varepsilon_0(15q_0\lambda_0(2\varepsilon_0 + 7\lambda_0^2 - 3\psi_0) + q_1(-8\varepsilon_0 + 9(-5\lambda_0^2 + \psi_0)))/5!,$

$q_6 = \varepsilon_0(30q_1\lambda_0(5\varepsilon_0 + 14\lambda_0^2 - 6\psi_0) - q_0(22\varepsilon_0^2 + 6\varepsilon_0(70\lambda_0^2 - 11\psi_0) + 45(21\lambda_0^4 - 14\lambda_0^2\psi_0 + \psi_0^2)))/6!$

$q_7 = (\varepsilon_0(-q_1(172\varepsilon_0^2+36\varepsilon_0(70\lambda_0^2-11\psi_0)+225(21\lambda_0^4-14\lambda_0^2\psi_0+\psi_0^2))+63q_0\lambda_0(12\varepsilon_0^2+4\varepsilon_0(25\lambda_0^2-9\psi_0)+5(33\lambda_0^4-30\lambda_0^2\psi_0+5\psi_0^2))))/7!$,

$q_8 = (\varepsilon_0(126q_1\lambda_0(52\varepsilon_0^2+14\varepsilon_0(25\lambda_0^2-9\psi_0)+15(33\lambda-064-30\lambda-0^2\psi-0+5\psi_0^2))-q_0(584\varepsilon_0^3+36\varepsilon_0^2(560\lambda_0^2-73\psi_0)+54\varepsilon_0(1925\lambda_0^4-1120\lambda_0^2\psi_0+67\psi_0^2)+315(429\lambda_0^6-495\lambda_0^4\psi_0+135\lambda_0^2\psi_0^2-5\psi_0^3))))/8!$

$q_9 = (\varepsilon_0(15q_0\lambda_0(2368\varepsilon-0^3+444\varepsilon_0^2(77\lambda_0^2-24\psi_0)+18\varepsilon_0(7007\lambda_0^4-5698\lambda_0^2\psi_0^2+827\psi_0^2)+189(715\lambda_0^6-1001\lambda_0^4\psi_0+385\lambda_0^2\psi_0^2-35\psi_0^3))-q_1(7136\varepsilon_0^3+108\varepsilon_0^2(1785\lambda_0^2-232\psi_0)+432\varepsilon_0(1925\lambda_0^4-1120\lambda_0^2\psi_0^2+67\psi_0^2)+2205(429\lambda_0^6-495\lambda_0^4\psi_0+135\lambda_0^2\psi_0^2-5\psi_0^3))))/9!$,

$q_{10} = (\varepsilon_0(30q_1\lambda_0(15220\varepsilon_0^3+12\varepsilon_0^2(14938\lambda_0^2-4647\psi_0)+81\varepsilon_0(7007\lambda_0^4-5698\lambda_0^2\psi_0+827\psi_0^2)+756(715\lambda_0^6-1001\lambda_0^4\psi_0+385\lambda_0^2\psi_0^2-35\psi_0^3))-q_0(28384\varepsilon_0^4+48\varepsilon_0^3(31735\lambda_0^2-3548\psi_0)+54\varepsilon_0^2(245245\lambda_0^4-126940\lambda_0^2\psi_0+6559\psi_0^2)+90\varepsilon_0(420420\lambda_0^6-441441\lambda_0^4\psi_0+107514\lambda_0^2\psi_0^2-3461\psi_0^3)+14175(2431\lambda_0^8-4004\lambda_0^6\psi_0+2002\lambda_0^4\psi_0^2-308\lambda_0^2\psi_0^3+7\psi_0^4))))/10!$.

УНИВЕРЗАЛНИ СИМБОЛИЧКИ ИЗРАЗ ЗА РАДИЈАЛНУ УДАЉЕНОСТ ПУТАЊЕ ДАТЕ КОНУСНИМ ПРЕСЕКОМ

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Оригинални научни рад

У овом раду изведен је универзални симболички израз за радијалну удаљеност путање дате конусним пресеком. Значај овог аналитичког израза је у томе што је он *инваријантан* у односу на многе операције које један степени ред преводе у други: сабирање, множење, степеновање, интеграљење, диференцирање, итд. Ово даје велику флексибилност у раду на аналитичким и нумеричким аспектима проблема који се баве радијалном удаљеношћу. Што се тиче нумеричког дела, у раду је представљен рекурзивни алгоритам за израчунавање коефицијената чланова реда. Коришћењем теорије верижних разломака дат је ефикасан метод за развој у ред и предложене су две технике које обезбеђују конвергенцију у случајевима када је интервал $(t-t_0)$ велики. Алгоритам, додатно, не захтева решавање Кеплерове једначине или аналогних једначина за параболичке и хиперболичке путање. Нумерички алгоритам је примењен на три орбите различитих ексцентрицитета и резултати су показали његову тачност, без обзира на врсту конусног пресека.