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COLLISION DYNAMICS IN HÉNON-HEILES' TWO-BODY PROBLEM

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SUMMARY: We tackle the two-body problem associated to Hénon-Heiles' potential in the special case of the collision singularity. Using McGehee-type transformations of the second kind, we blow up the singularity and replace it by the collision manifold M_c pasted on the phase spece. We fully describe the flow on M_c. This flow is similar to analogous flows met in post-Newtonian two-body problems.

Key words. Celestial mechanics

1. INTRODUCTION

Hénon and Heiles' (1964) potential was introduced in order to model the motion of a star within a galaxy. This potential has the form

$$U(x, y) = Ax^2 + By^2 + Cx^2y + Dy^3, \qquad (1)$$

with $A, B, C, D \in \mathbf{R}$, $A, B > 0$. Originally it was examined for the case of axial symmetry, whereas in this paper one applies it to the case of the planar motion of a test particle (star) in the plane $z = 0$. For this reason, as will see further on, the angular-momentum integral does not exist.

Hénon-Heiles' potential was constructed by adding two terms of third degree in coordinates to the potential of a planar harmonic oscillator. It also results from expansion of the potential corresponding to an integrable system (resulting from some canonical transformations applied to a system modelling the motion of three particles on a circle under the influence of exponentially decreasing forces) to third-degree terms (see Boccaletti and Pucacco 1996, Anisiu and Pál 1999).

In this paper we tackle the collision dynamics in the two-body problem associated to Hénon-Heiles' model. In Section 2 we establish equations of the motion in configuration-momentum coordinates, as well as the existence of the first integral of energy. We also show that the angular momentum is not conserved.

In Section 3 we transform the equations of motion and the energy integral into standard polar coordinates. The equations of motion represent singularities corresponding to both collision and escape. We deal here only with collision singularities, and blow them up via a Sundman-type transformation of the time. In fact, both the change from configuration-momentum coordinates to standard polar coordinates and the time rescaling constitute steps of the McGehee-type second-kind transformations (McGehee 1974), which provide regularized equations of motion (Section 4).

Under the McGehee-type transformations, the collision singularity was blown up and replaced by a manifold M_c pasted on the phase space. Section 5 describes this manifold (which is a 2D cylinder or a 2D torus) and the flow on it. Even if this flow is deprived of physical significance, it provides valuable information about the orbits that approach collision (due to the continuity of solutions with respect to initial conditions).

In spite of the fact that Hénon-Heiles' model is anisotropic, the collision manifold is fairly simple: a torus with an upper circle (UC) and a lower circle (LC) of degenerate equilibria, and heteroclinic orbits that move from LC to UC (for comparison with other anisotropic fields, see Gutzwiller 1973, Devaney 1978, Craig et al. 1999, Mioc et al. 2003).

2. BASIC EQUATIONS

Let us consider the motion of a particle of unit mass with respect to the field-generating source. Let $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2$ be the position (configuration) vector of the particle. The potential (1) will read

$$U(\mathbf{q}) = Aq_1^2 + Bq_2^2 + Cq_1^2 q_2 + Dq_2^3. \tag{2}$$

Denote by $T(\mathbf{p}) = |\mathbf{p}|^2/2$ the kinetic energy of the particle, where $\mathbf{p}(= \dot{\mathbf{q}}) = (p_1, p_2) \in \mathbf{R}^2$ is the momentum vector. The equations of motion read

$$\begin{aligned} \dot{\mathbf{q}} &= \partial H(\mathbf{q},\mathbf{p})/\partial \mathbf{p}, \\ \dot{\mathbf{p}} &= -\partial H(\mathbf{q},\mathbf{p})/\partial \mathbf{q}, \end{aligned} \tag{3}$$

in which the Hamiltonian has the expression $H(\mathbf{q},\mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q})$, or, explicitly,

$$H(\mathbf{q},\mathbf{p}) = (p_1^2+p_2^2)/2 - Aq_1^2 - Bq_2^2 - Cq_1^2 q_2 - Dq_2^3. \tag{4}$$

By (3) and (4), we get the equations of motion in the form

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= 2Aq_1 + 2Cq_1 q_2, \\ \dot{p}_2 &= Cq_1^2 + 2Bq_2 + 3Dq_2^2. \end{aligned} \tag{5}$$

It is clear that the Hamiltonian is a constant of motion, namely

$$H(\mathbf{q},\mathbf{p}) = h, \tag{6}$$

which provides the first integral of energy: along each orbit the total energy of the system is conserved. In (6) h is the energy constant.

Defining the angular momentum $L(\mathbf{q},\mathbf{p}) = q_1 p_2 - q_2 p_1$, it is also clear that $\dot{L} \equiv 0$ only for $A = B$ and $C = 0 = D$. In other words, the angular momentum is not conserved, so we do not have a corresponding first integral. This was to be expected, given the anisotropic structure of Hénon-Heiles' potential.

3. POLAR COORDINATES

Since our problem is anisotropic, it is more convenient to tackle it in polar coordinates (r, θ). Let us define the real analytic diffeomorphism $\mathbf{R}^4 \to \mathbf{R} \times S^1 \times \mathbf{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (r, \theta, u, v)$ via the transformations

$$\begin{aligned} &r = |\mathbf{q}|, \quad \theta = \arctan(q_2/q_1), \\ &u = \dot{r} = (q_1 p_1 + q_2 p_2)/|\mathbf{q}|, \\ &v = r\dot{\theta} = (q_1 p_2 - q_2 p_1)/|\mathbf{q}|. \end{aligned} \tag{7}$$

Under these transformations (which also represent the first step of McGehee (1974)-type transformations of the second kind), the equations of motion (5) become

$$\begin{aligned} &\dot{r} = u, \quad \dot{\theta} = v/r \\ &\dot{u} = v^2/r + (2A + 3Cr\sin\theta) r\cos^2\theta + \\ &+(2B + 3Dr\sin\theta) r\sin^2\theta, \\ &\dot{v} = -uv/r + 2(B-A) r\sin\theta\cos\theta + \\ &+[3(D-C)\sin^2\theta + C] r^2\cos\theta. \end{aligned} \tag{8}$$

whereas the energy integral (6) acquires the form

$$\begin{aligned} &(u^2+v^2)/2 - (A\cos^2\theta + B\sin^2\theta) r^2 - \\ &-(C\cos^2\theta + D\sin^2 theta) r^3 \sin\theta = h. \end{aligned} \tag{9}$$

Note that, in (7), u and v are the standard polar components of the velocity.

4. REGULAR EQUATIONS OF MOTION

Examining the equations of motion (8), we see that they are singular for $r \to 0$. This situation corresponds to a collision of the particle with the field-generating source. To prove it, an imitation of Painlevé's criterion is sufficient.

Consider hence the collision $r = 0$. It is interesting that the collision is not a singularity in configuration-momentum coordinates $(\mathbf{q},\mathbf{p})$, but is a singularity in polar coordinates. Also, the energy integral is regular in both configuration-momentum and polar coordinates. (However, there are restrictions for the energy integral in this case, but they will be discussed later.)

To remove the singularity in Eqs. (8), we resort to a Sundman-type transformation

$$ds = r^{-1} dt, \tag{10}$$

which rescales the time. (Such a change of variable constitutes the last step of McGehee transformations of the second kind.) Introducing (10), the motion equations become

$$\begin{aligned} &r' = ru, \quad \theta' = v, \\ &u' = v^2 + (2A + 3Cr\sin\theta) r^2\cos^2\theta + \\ &+(2B + 3Dr\sin\theta) r^2\sin^2\theta, \\ &v' = -uv + 2(B-A) r^2\sin\theta\cos\theta + \\ &+[3(D-C)\sin^2\theta + C] r^3\cos\theta, \end{aligned} \tag{11}$$

where $(\cdot)' = d(\cdot)/ds$, and we kept, by abuse, the same notation for the new functions of the timelike variable s. Of course, the energy integral (9) remains the same.

5. COLLISION MANIFOLD

The equations of motion (11) are well-defined for the boundary $r = 0$. They extend smoothly to this boundary, which is invariant to the flow, because, from the first Eq. (11), $r' = 0$ for $r = 0$.

In this way, we have blown up the singularity $r = 0$ and replaced it by a manifold pasted on the phase space. In what follows we shall describe this manifold.

Let us first define the $(r = 0)$-manifold:

$$M_0 = \{(r, \theta, u, v)|\ r = 0,\ \theta \in S^1,\ (u, v) \in \mathbf{R}^2\}, \tag{12}$$

and the constant-energy manifold:

$$M_h = \{(r, \theta, u, v)|\ r \in [0, \infty),\ \theta \in S^1,\ (9) \text{ holds}\}. \tag{13}$$

Now, we define the *collision manifold* M_c as

$$\begin{aligned} M_c &= M_0 \cap M_h = \\ &= \{(r, \theta, u, v)|\ r = 0,\ \theta \in S^1,\ u^2 + v^2 = 2h\}. \end{aligned} \tag{14}$$

It is clear that M_c represents a 2D cylinder of radius $\sqrt{2h}$. But, since S^1 is the segment $[0, 2\pi]$ with the ends pasted together, the cylinder may be assimilated to a 2D torus. Both the cylinder and the torus are embedded in the 3D space of the coordinates (θ, u, v), embedded, in turn, in the 4D space of the McGehee coordinates (r, θ, u, v).

A very important issue is to be emphasized here. By (14), for $h > 0$, M_c is a nondegenerate cylinder or torus; for $h = 0$, M_c reduces to a segment of length 2π, or to a circle; for $h < 0$, M_c is an empty set. In physical terms, this means that collisions are possible only for $h \geq 0$. For $h > 0$, the particle collides with the field-source with positive velocity. For $h = 0$, the particle collides with the field-source with zero velocity. For $h < 0$, there are no collisions.

By (10), another important issue is to be remarked. The particle needs an infinite amount of fictitious time s to reach the collision with the field source. This means that M_c is a manifold of equilibria for the global flow on the full phase-space of the coordinates (r, θ, u, v). In turn, this means that the collision velocities derived above (positive for $h > 0$, zero for $h = 0$) are asymptotic velocities within the timescale defined by (10).

Now, the collision manifold being geometrically represented, let us depict the flow on it for $h > 0$. This flow is deprived of physical significance, but – due to the continuity of solutions with respect to initial conditions – it provides valuable information about orbits that approach collision.

Putting $r = 0$ in (11), we get the vector field on M_c:

$$\begin{aligned} \theta' &= v, \\ u' &= v^2, \\ v' &= -uv. \end{aligned} \tag{15}$$

One immediately sees, by the second Eq. (15), that the flow on M_c is gradientlike with respect to the u-coordinate (in other words, except equilibria, every solution increases monotonically on this coordinate).

The vector field (15) shows that the flow on the torus M_c has two circles of degenerate equilibria: the upper circle

$$UC = \{(\theta, u, v)|\ \theta = \theta_0,\ u = \sqrt{2h},\ v = 0\}, \tag{16}$$

and the lower circle

$$LC = \{(\theta, u, v)|\ \theta = \theta_0,\ u = -\sqrt{2h},\ v = 0\}, \tag{17}$$

with arbitrary $\theta_0 \in S^1$.

Given the gradientlikeness of the flow on M_c, all other orbits are heteroclinic and move from LC to UC. To determine the slope of these curves, let us put $u = \sqrt{2h} \sin\alpha$ and $v = -\sqrt{2h}\cos\alpha$. This yields straightforwardly $d\alpha/d\theta = -1$. The flow on M_c (considered as a cylinder) is plotted in Fig. 1.

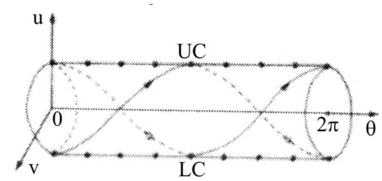

Fig. 1. *The collision manifold as a cylinder and the flow on it.*

Taking into account the first Eq. (11), the halfplane $u < 0$ corresponds to orbits that approach collision ($r \to 0$ in the future), while the halfplane $u > 0$ corresponds to ejection orbits ($r \to 0$ in the past). Considering the flow on M_c (Fig. 1), this means that collision solutions are regularizable.

6. CONCLUDING REMARKS

We would like to emphasize that the fictitious flow on the collision manifold in Hénon-Heiles' two-body problem is similar to such flows met in problems associated to quasihomogeneous potentials (Mioc and Stavinschi 2001). Moreover, it is identical to the fictitious flows on the infinity manifold ($r \to \infty$) in two-body problems associated with the certain post-Newtonian fields (relativistic or not), as, for instance, Fock's (Mioc and Stavinschi 2000) or Mücket-Treder's (Mioc 2002) ones.

In addition, let us remark that, even associated to an anisotropic field, Hénon-Heiles' two-body problem exhibits a fairly simple collision manifold (cf. Gutzwiller 1973, Devaney 1978, Craig et al. 1999, Mioc et al. 2003). As we shall show elsewhere, the corresponding infinity manifold has a much more intricate structure.

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КОЛИЗИОНА ДИНАМИКА У ЕНОН-ХЕЈЛЕСОВОМ ПРОБЛЕМУ ДВА ТЕЛА

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Оригинални научни рад

Аутори проучавају проблем два тела за Енон-Хејлесов потенцијал у специјалном случају колизионе сингуларности. Користећи трансформације МекГијевог типа друге врсте, сингуларност се одстрањује и замењује колизионим мноштвом M_c за фазни простор. У потпуности је описан доток на M_c. Овај доток је сличан аналогним дотоцима који се срећу у пост-Њутновским проблемима двају тела.